MONOTONICITY PROPERTIES OF GAUSSIAN HYPERGEOMETRIC FUNCTIONS WITH RESPECT TO THE PARAMETER

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Abstract. The authors establish the necessary and sufficient conditions under which certain combinations of Gaussian hypergeometric function and elementary function are monotone in the parameter, which generalize the recent results of generalized elliptic integrals of the first and second kinds obtained by Qiu et al. Moreover, the authors also prove two monotonicity theorems of generalized elliptic integrals from another point of view.

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