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ON PROPERTIES OF WEIGHTED HARDY CONSTANT FOR MEANS

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Abstract. For a given weighted mean $\mathscr{M}$ defined on a subinterval of $\mathbb{R}_+$ and a sequence of weights $\lambda = (\lambda_n)_{n=1}^{\infty}$ we define a Hardy constant $\mathscr{H}(\lambda)$ as the smallest extended real number such that

$$\sum_{n=1}^{\infty} \lambda_n \mathscr{M}\big((x_1,\dots,x_n),(\lambda_1,\dots,\lambda_n)\big) \leqslant \mathscr{H}(\lambda) \cdot \sum_{n=1}^{\infty} \lambda_n x_n \text{ for all } x \in \ell^1(\lambda).$$

The aim of this note is to present a comprehensive study of the mapping $\mathscr{H}$. For example we prove that it is lower semicontinuous in the pointwise topology.

Moreover we show that whenever $\mathscr{M}$ is a monotone and Jensen-concave mean which is continuous in its weights then $\mathscr{H}$ is monotone with respect to the partitioning of the vector. Finally we deliver some sufficient conditions for λ to validate the equality $\mathscr{H}(\lambda) = \sup \mathscr{H}$ for every symmetric and monotone mean.

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