INTEGRAL EQUATIONS ON COMPACT MANIFOLD WITH BOUNDARY

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Abstract. Let $(M^n, g, \Sigma)$ be a smooth compact Riemannian manifold with boundary and $n \geq 3$. This paper is devoted to studying a class of integral system
\[
\begin{cases}
g^{\alpha - 1} f (x) = \int_{\Sigma} K(x, y) f(y) dS_y, & x \in M^n, \\
f^{\beta - 1}(y) = \int_{M^n} K(x, y) g(x) dV_x, & y \in \Sigma,
\end{cases}
\]
where $\alpha \in (1, n)$, $\rho = \frac{2\alpha}{n+\alpha}$, $\tilde{\rho} = \frac{2(n-1)}{n+\alpha-2}$, $(f, g) \in L^{\rho_1}(\Sigma) \times L^{\rho_2}(M^n)$ and the kernel function
\[
K(x, y) \in C_{c}^{\infty}(M^n \times M^n \setminus \{(x, x)\})
\]
satisfies $K(x, y) \sim |x - y|^{\alpha - n}$ as $|x - y|_g \to 0$. Since the system is the Euler-Lagrange equations of extremal problem
\[
N_k(\alpha, M) = \sup \left\{ \left( \int_{M^n} \int_{\Sigma} g(x) K(x, y) f(y) dS_y dV_x : \| f \|_{L^{\rho_1}(\Sigma)} = \| g \|_{L^{\rho_2}(M^n)} = 1 \right) \right\}
\]
we will study the existence of the system by concentration-compactness principle. Firstly, we get $N_k(\alpha, M) \geq C_c(n, \alpha, \tilde{\rho}_\alpha)$, where $C_c(n, \alpha, \tilde{\rho}_\alpha)$ is the best constant of Hardy-Littlewood-Sobolev inequalities on the upper half space established by Dou and Zhu [6] and equals to $N_k(\alpha, M)$ when $(M^n, g, \Sigma) = (B_1(0), | \cdot |, \partial B_1(0))$ and $K(x, y) = |x - y|^{\alpha - n}$. Secondly, if $N_k(\alpha, M) > C_c(n, \alpha, \tilde{\rho}_\alpha)$, we prove that $N_k(\alpha, M)$ is attained. Namely, under the criterion $N_k(\alpha, M) > C_c(n, \alpha, \tilde{\rho}_\alpha)$, we get the existence of the system. Lastly, a concrete example satisfying the criterion is given. The example is closely related to the conformal problems studied by Escobar [9, 10].


Keywords and phrases: Integral equations, concentration-compactness principle, compact Riemannian manifold with boundary, Hardy-Littlewood-Sobolev inequality.

REFERENCES


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