

## THE CONTINUITY OF PSEUDO-DIFFERENTIAL OPERATORS ON LOCAL VARIABLE HARDY SPACES AND THEIR DUAL SPACES

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*Abstract.* In this paper we establish the continuity of pseudo-differential operators with symbols in  $S_{1,\delta}^{-\alpha}$  on local variable Hardy spaces and their dual spaces. Precisely, we show that the pseudo-differential operator maps continuously one local variable Hardy space into another one, maps continuously one local variable Carleson measure spaces into another one and maps continuously one variable Lebesgue spaces into one local variable Carleson measure spaces.

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