Mathematical Inequalities & Applications
Volume 27, Number 1 (2024), 193–200
doi:10.7153/mia-2024-27-15

ON A CONJECTURE RELATED TO THE GEOMETRIC MEAN AND NORM INEQUALITIES

SHAIMA'A FREEWAN AND MOSTAFA HAYAJNEH*

Abstract. A conjecture of Dinh, Ahsani, and Tam, was recently settled in [7]. In this note, we give a refinement to that result, namely if A_i and B_i are positive definite matrices and $Z=[Z_{ij}]$ is the block matrix such that $Z_{ij}=B_i^{\frac{1}{2}}\left(\sum_{k=1}^{m}A_k\right)B_j^{\frac{1}{2}}$ for all $i,j=1,\cdots,m$, then

$$\left|\left|\left|\sum_{i=1}^{m}\left(A_i^2\sharp B_i^2\right)^r\right|\right|\right| \leqslant |||Z^r||| \leqslant \left|\left|\left|\left(\left(\sum_{i=1}^{m}A_i\right)^{\frac{rp}{2}}\left(\sum_{i=1}^{m}B_i\right)^{rp}\left(\sum_{i=1}^{m}A_i\right)^{\frac{rp}{2}}\right)^{\frac{1}{p}}\right|\right|\right|,$$

for all unitarily invariant norms, for all $p>0$ and $r\geqslant 1$ such that $rp\geqslant 1$. Our approach provides us with an alternative proof without using the method of majorization that was used in [7]. As a byproduct, we get a refinement to a result of Audenaert in 2015.

Mathematics subject classification (2020): Primary 15A60; Secondary 15B57, 47A30, 47B15.
Keywords and phrases: Bourin question, unitarily invariant norm, positive definite matrix, inequality.

REFERENCES

[1] T. H. DINH, S. AHSANI, AND T.Y. TAM, *Geometry and inequalities of geometric mean*, Czechoslovak Math. J., **66**, 3 (2016), 777–792.

[2] K. AUDENAERT, *A norm inequality for pairs of commuting positive semidefinite matrices*, Electronic Journal of Linear Algebra, **30**, (2015), 80–84.

[3] R. BHATIA, *Matrix Analysis*, Springer-Verlag, New York, 1997.

[4] R. BHATIA, *Positive Definite Matrices*, Princeton University Press, Princeton, 2007.

[5] J. C. BOURIN, *Matrix subadditivity inequalities and block-matrices*, Internat. J. Math., **20**, 6 (2009), 679–691.

[6] J. C. BOURIN AND M. UCHIYAMA, *A matrix subadditivity inequality for $f(A+B)$ and $f(A)+f(B)$*, Linear Algebra Appl., **423**, 2–3 (2007), 512–518.

[7] S. FREEWAN AND M. HAYAJNEH, *On norm inequalities related to the geometric mean*, Linear Algebra Appl., **670**, (2023), 104–120.

[8] S. HAYAJNEH AND F. KITTANEH, *Trace inequalities and a question of Bourin*, Bull. Aust. Math., **88**, 3 (2013), 384–389.

© ELEMENT, Zagreb
Paper MIA-27-15