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COMPLETE MONOTONICITY OF THE REMAINDER OF AN ASYMPTOTIC EXPANSION OF THE GENERALIZED GURLAND'S RATIO

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Abstract. Let $a,b,c,d \in \mathbb{R}$ with $a+b=c+d=2r+1$. Then

$$\ln\frac{\Gamma(x+a)\Gamma(x+b)}{\Gamma(x+c)\Gamma(x+d)} \sim \sum_{k=1}^{\infty} \frac{B_{2k}(\theta_1)-B_{2k}(\theta_2)}{k(2k-1)(x+r)^{2k-1}} \text{ as } x\to\infty,$$

where $(\delta_1,\delta_2)=(|a-b|,|c-d|)=(1-2\theta_1,1-2\theta_2)$. When $0\leqslant\delta_2<\delta_1\leqslant 1$, the function

$$x\mapsto(-1)^m\left[\ln\frac{\Gamma(x+a)\Gamma(x+b)}{\Gamma(x+c)\Gamma(x+d)}-\sum_{k=1}^{m}\frac{B_{2k}(\theta_1)-B_{2k}(\theta_2)}{k(2k-1)(x+r)^{2k-1}}\right]$$

for $m\in\mathbb{N}$ is completely monotonic on $(-r,\infty)$. This yields some known and new results.

Mathematics subject classification (2020): 41A60, 33B15, 26A48, 26D15.
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