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SHARP WEIGHTED HÖLDER MEAN BOUNDS FOR SEIFFERT'S MEANS

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Abstract. Let $P(a,b)$ and $T(a,b)$ be the first and second Seiffert's means for two positive numbers a and b, in this paper, for any fixed $p \in \mathbb{R}$, we present the optimal parameters $\alpha_p, \beta_p, \lambda_p, \mu_p \in [0,1]$ such that the inequalities

$$H_p(a,b;\alpha_p) \leqslant P(a,b) \leqslant H_p(a,b;\beta_p), \qquad H_p(a,b;\lambda_p) \leqslant T(a,b) \leqslant H_p(a,b;\mu_p)$$

hold true for all $a,b>0$, where $H_p(a,b;\omega)$ is the weighted p-order Hölder (power) mean with the weight $\omega \in [0,1]$. As applications, various sharp inequalities for $P(a,b)$ and $T(a,b)$, including the sharp power mean bounds, will be established.

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