GENERALIZED AMOS–TYPE BOUNDS FOR
MODIFIED BESSEL FUNCTION RATIOS

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Abstract. Amos-type and generalized Amos-type bounds have been established in the literature as lower and upper bounds for the modified Bessel function ratio $R_\nu(t) = I_{\nu+1}(t)/I_\nu(t)$ for $t > 0$. We complement previous results by providing a family of improved explicit lower bounds of the generalized Amos-type given by $G_{\alpha,\beta,\lambda}(t) = t/(\alpha + \sqrt{\lambda t^2 + \beta^2})$. We show that the difference of two such bounds has a single sign change, and that for every $t > 0$ the optimal bound can easily be determined. We also show that the upper bound for the modified Bessel function ratio established by Amos cannot be improved by considering $\lambda > 0$ instead of fixing $\lambda = 1$.

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REFERENCES

[1] D. E. AMOS, *Computation of Modified Bessel Functions and Their Ratios*, Mathematics of Computation, **28** (1974), pp. 239–251.

[2] Á. BARICZ AND E. NEUMAN, *Inequalities Involving Modified Bessel Functions of the First Kind II*, Journal of Mathematical Analysis and Applications, **332** (2007), pp. 265–271.

[3] K. HORNIK AND B. GRÜN, *Amos-Type Bounds for Modified Bessel Function Ratios*, Journal of Mathematical Analysis and Applications, **408** (2013), pp. 91–101.

[4] A. LAFORGIA AND P. NATALINI, *Some Inequalities for Modified Bessel Functions*, Journal of Inequalities and Applications, **2010** (2010).

[5] I. NÅSELL, *Rational Bounds for Ratios of Modified Bessel Functions*, SIAM Journal on Mathematical Analysis, **9** (1978), pp. 1–11.

[6] E. NEUMAN, *Inequalities Involving Modified Bessel Functions of the First Kind*, Journal of Mathematical Analysis and Applications, **171** (1992), pp. 532–536.

[7] G. SCHOU, *Estimation of the Concentration Parameter in von Mises-Fisher Distributions*, Biometrika, **65** (1978), pp. 369–377.

[8] J. SEGURA, *Bounds for Ratios of Modified Bessel Functions and Associated Turán-type Inequalities*, Journal of Mathematical Analysis and Applications, **374** (2011), pp. 516–528.

[9] H. C. SIMPSON AND S. J. SPECTOR, *On Barrelling for a Special Material in Finite Elasticity*, Quarterly of Applied Mathematics, **42** (1984), pp. 99–111.

[10] H. C. SIMPSON AND S. J. SPECTOR, *Some Monotonicity Results for Ratios of Modified Bessel Functions*, Quarterly of Applied Mathematics, **42** (1984), pp. 95–98.

[11] Z.-H. YANG AND S.-Z. ZHENG, *Monotonicity and Convexity of the Ratios of the First Kind Modified Bessel Functions and Applications*, Mathematical Inequalities and Applications, **21** (2018), pp. 107–125.

[12] Z.-H. YANG AND S.-Z. ZHENG, *Monotonicity of the Ratio of Modified Bessel Functions of the First Kind with Applications*, Journal of Inequalities and Applications, **2018** (2018), p. 57.

[13] L. YUAN AND J. D. KALBFLEISCH, *On the Bessel Distribution and Related Problems*, The Annals of the Institute of Statistical Mathematics, **52** (2000), pp. 438–447.

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