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QUADRUPLE INEQUALITIES: BETWEEN CAUCHY–SCHWARZ AND TRIANGLE

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Abstract. We prove a set of inequalities that interpolate the Cauchy–Schwarz inequality and the triangle inequality. Every nondecreasing, convex function with a concave derivative induces such an inequality. They hold in any metric space that satisfies a metric version of the Cauchy–Schwarz inequality, including all CAT(0) spaces and, in particular, all Euclidean spaces. Because these inequalities establish relations between the six distances of four points, we call them quadruple inequalities. In this context, we introduce the quadruple constant — a real number that quantifies the distortion of the Cauchy–Schwarz inequality by a given function. Additionally, for inner product spaces, we prove an alternative, more symmetric version of the quadruple inequalities, which generalizes the parallelogram law.

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REFERENCES

[1] GEROLD ALSMEYER AND UWE RÖSLER, *The best constant in the Topchii-Vatutin inequality for martingales*, Statist. Probab. Lett., **65** (3): 199–206, 2003.
[2] I. D. BERG AND I. G. NIKOLAEV, *Quasilinearization and curvature of Aleksandrov spaces*, Geom. Dedicata, **133**: 195–218, 2008.
[3] LEONARD M. BLUMENTHAL, *Theory and applications of distance geometry*, Chelsea Publishing Co., New York, second edition, 1970.
[4] MARTIN R. BRIDSON AND ANDRÉ HAEFLIGER, *Metric spaces of non-positive curvature*, vol. **319** of Grundlehren der mathematischen Wissenschaften, Springer-Verlag, Berlin, 1999.
[5] S. M. BUCKLEY, K. FALK AND D. J. WRAITH, *Ptolemaic spaces and CAT(0)*, Glasg. Math. J., **51** (2): 301–314, 2009.
[6] P. CHARBONNIER, L. BLANC-FERAUD, G. AUBERT AND M. BARLAUD, *Two deterministic half-quadratic regularization algorithms for computed imaging*, in Proceedings of 1st International Conference on Image Processing, vol. **2**, pages 168, 169, 170, 171, 172, Los Alamitos, CA, USA, 11 1994, IEEE Computer Society.
[7] PAUL CORAZZA, *Introduction to metric-preserving functions*, Amer. Math. Monthly, **106** (4): 309–323, 1999.
[8] MICHEL MARIE DEZA AND ELENA DEZA, *Encyclopedia of distances*, Springer, Berlin, fourth edition, 2016.
[9] PER ENFLO, *On the nonexistence of uniform homeomorphisms between L_p-spaces*, Ark. Mat., **8**: 103–105, 1969.
[10] PER ENFLO, *Uniform structures and square roots in topological groups. II*, Israel J. Math., **8**: 253–272, 1970.
[11] TIMOTHY FAVER, KATELYNN KOCHALSKI, MATHAV KISHORE MURUGAN, HEIDI VERHEGGEN, ELIZABETH WESSON AND ANTHONY WESTON, *Roundness properties of ultrametric spaces*, Glasg. Math. J., **56** (3): 519–535, 2014.

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[12] P. THOMAS FLETCHER, SURESH VENKATASUBRAMANIAN AND SARANG JOSHI, *The geometric median on Riemannian manifolds with application to robust atlas estimation*, NeuroImage, **45** (1, Supplement 1): S143–S152, 2009, Mathematics in Brain Imaging.
[13] THOMAS FOERTSCH, ALEXANDER LYTCHAK AND VIKTOR SCHROEDER, *Nonpositive curvature and the Ptolemy inequality*, Int. Math. Res. Not. IMRN, (22): Art. ID rnm100, 15, 2007.
[14] MAURICE FRÉCHET, *Les éléments aléatoires de nature quelconque dans un espace distancié*, Ann. Inst. H. Poincaré, **10**: 215–310, 1948.
[15] PETER J. GREEN, *Bayesian reconstructions from emission tomography data using a modified EM algorithm*, IEEE transactions on medical imaging, **9** 1: 84–93, 1990.
[16] PETER J. HUBER, *Robust estimation of a location parameter*, Ann. Math. Statist., **35**: 73–101, 1964.
[17] JOVAN KARAMATA, *Sur une inégalité rélative aux fonctions convexes*, Publ. Math. Univ. Belgrade, **1**: 145–148, 1932.
[18] JU. G. REŠETNJAK, *Non-expansive maps in a space of curvature no greater than K*, Sibirsk. Mat. Ž., pages 918–927, 1968.
[19] CHRISTOF SCHÖTZ, *Convergence rates for the generalized Fréchet mean via the quadruple inequality*, Electron. J. Stat., **13** (2): 4280–4345, 2019.
[20] CHRISTOF SCHÖTZ, *Strong laws of large numbers for generalizations of Fréchet mean sets*, Statistics, **56** (1): 34–52, 2022.
[21] CHRISTOF SCHÖTZ, *Quadruple Inequalities: Between Cauchy-Schwarz and Triangle*, 2024, https://arxiv.org/abs/2307.01361.
[22] J. MICHAEL STEELE, *The Cauchy-Schwarz master class*, AMS/MAA Problem Books Series, Mathematical Association of America, Washington, DC; Cambridge University Press, Cambridge, 2004, An introduction to the art of mathematical inequalities
[23] KARL-THEODOR STURM, *Probability measures on metric spaces of nonpositive curvature*, in Heat kernels and analysis on manifolds, graphs, and metric spaces (Paris, 2002), vol. **338** of Contemp. Math., pages 357–390, Amer. Math. Soc., Providence, RI, 2003.
[24] V. A. TOPCHIJ AND V. A. VATUTIN, *Maximum of the critical Galton-Watson processes and left-continuous random walks*, Teor. Veroyatn. Primen., **42** (1): 21–34, 1997.