REVERSED HARDY–LITTLEWOOD–SOBOLEV INEQUALITY
ON HEISENBERG GROUP $\mathbb{H}^n$ AND CR SPHERE $\mathbb{S}^{2n+1}$

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Abstract. This paper is mainly devoted to the study of the reversed Hardy-Littlewood-Sobolev (HLS) inequality on Heisenberg group $\mathbb{H}^n$ and CR sphere $\mathbb{S}^{2n+1}$. First, we establish the roughly reversed HLS inequality and give an explicitly lower bound for the sharp constant. Then, the existence of the extremal functions with sharp constant is proved by *subcritical approach* and some compactness techniques. Our method is *rearrangement free* and can be applied to study the classical HLS inequality and other similar inequalities.

Mathematics subject classification (2020): 30C70, 45E10, 26D10.

Keywords and phrases: Heisenberg group, reversed Hardy-Littlewood-Sobolev inequality, subcritical approach, rearrangement free method.

REFERENCES

[1] W. BECKNER, *Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality*, Ann. of Math., **138** (1993), 213–242.

[2] W. BECKNER, *Functionals for Multilinear Fractional Embedding*, Acta Math. Sinica, English Series, **31** (2015), 1–28.

[3] T. P. BRANSON, L. FONTANA, C. MORPURGO, *Moser-Trudinger and Beckner-Onofri's inequalities on the CR sphere*, Ann. of Math., **177** (2013), 1–52.

[4] I. BIRINDELLI, J. PRAJAPAT, *Nonlinear Liouville theorems in the Heisenberg group via the moving plane method*, Comm. PDE., **24** (9&10) (1999), 1875–1890.

[5] E. A. CARLEN, M. LOSS, *Extremals of functionals with competing symmmetries*, J. Funct. Anal. **88** (2) (1990), 437–456.

[6] L. CHEN, Z. LIU, G. LU, C. TAO, *Reverse Stein-Weiss inequalities and existence of their extremal functions*, Trans. Amer. Math. Soc., **370** (12) (2018), 8429–8450.

[7] L. CHEN, G. LU, C. TAO, *Existence of extremal functions for the Stein-Weiss inequalities on the Heisenberg group*, J. Func. Anal., **277** (2019), 1112–1138.

[8] L. CHEN, G. LU, C. TAO, *Hardy-Littlewood-Sobolev inequalities with fractional Poisson kernel and their applications in PDEs*, Acta Math. Sin. (Engl. Ser.), **35** (6) (2019), 853–875.

[9] L. CHEN, G. LU, C. TAO, *Reverse Stein-Weiss inequalities on the upper half space and the existence of their extremals*, Adv. Nonlinear Stud., **19** (2019), 475–494.

[10] L. CHEN, Z. LIU, G. LU, C. TAO, *Stein-Weiss inequalities with fractional Poisson kernel*, Rev. Mat. Iberoam., **36** (5) (2020), 1289–1308.

[11] S. CHEN, *A new family of sharp conformally invariant integral inequalities*, Int. Math. Res. Not., **5** (2014), 1205–1220.

[12] W. CHEN, C. LI, B. OU, *Classification of solutions for an integral equation*, Comm. Pure Appl. Math., **59** (2006), 330–343.

[13] W. COHN, G. LU, *Best constants for Moser-Trudinger inequalities on the Heisenberg group*, Indiana Univ. Math. J., **50** (4) (2001), 1567–1591.

[14] W. COHN, G. LU, *Sharp constants for Moser-Trudinger inequalities on spheres in complex space $\mathbb{C}^n$*, Comm. Pure Appl. Math., **57** (2004), 1458–1493.

[15] J. DOU, *Weighted Hardy-Littlewood-Sobolev inequalities on the upper half space*, Commun. Contemp. Math., **18** (5) (2016), 1550067.

[16] J. DOU, Q. GUO, M. ZHU, *Negative power nonlinear integral equations on bounded domains*, J. Diff. Equ., **269** (2020), 10527–10557.

[17] J. DOU, Q. GUO, M. ZHU, *Subcritical approach to sharp Hardy-Littlewood-Sobolev type inequalities on the upper half space*, Adv. Math. **312** (2017), 1–45, 2017; Corrigendum to "Subcritical approach to sharp Hardy-Littlewood-Sobolev type inequalities on the upper half space" [Adv. Math. **312**: 1–45, 2017], Adv. Math. **317** (2017), 640–644.

[18] J. DOU, M. ZHU, *Nonlinear integral equations on bounded domains*, J. Funct. Anal., **277** (2019), 111–134.

[19] J. DOU, M. ZHU, *Reversed Hardy-Littewood-Sobolev inequality*, Int. Math. Res. Not., **19** (2015), 9696–9726.

[20] J. DOU, M. ZHU, *Sharp Hardy-Littlewood-Sobolev inequality on the upper half space*, Int. Math. Res. Not., **3** (2015), 651–687.

[21] S. DRAGOMIR, G. TOMASSINI, *Differential geometry and analysis on CR manifolds*, Birkhäuser, Boston, 2006.

[22] G. B. FOLLAND, E. M. STEIN, *Estimates for the $\overline{\partial}_b$ complex and analysis on the Heisenberg group*, Comm. Pure Appl. Math., **27** (1974), 429–522.

[23] G. B. FOLLAND, *Subelliptic estimates and function spaces on nilpotent Lie groups*, Arkiv för Matematik, **13** (1975), 161–207.

[24] G. B. FOLLAND, *A fundamental solution for a subelliptic operator*, Bull. Amer. Math. Soc., **79** (1973), 373–376.

[25] R. L. FRANK, E. H. LIEB, *Sharp constants in several inequalities on the Heisenberg group*, Ann. of Math., **176** (2012), 349–381.

[26] N. GAROFALO, E. LANCONELLI, *Frequency functions on the Heisenberg group, the uncertainty principle and unique continuation*, Ann. Inst. Fourier (Grenoble), **40** (1990), 313–356.

[27] M. GLUCK, *Subcritical approach to conformally invariant extension operators on the upper half space*, J. Funct. Anal., **278** (1) (2020), 1–46.

[28] L. GROSS, *Logarithmic Sobolev inequalities*, Amer. J. Math., **97** (1976), 1061–1083.

[29] Q. GUO, *Blowup analysis for integral equations on bounded domains*, J. Diff. Equ., **266** (2019), 8258–8280.

[30] X. HAN, *Existence of maximizers for Hardy-Littlewood-Sobolev inequalities on the Heisenberg group*, Indiana Univ. Math. J., **62** (3) (2013), 737–751.

[31] X. HAN, G. LU, J. ZHU, *Hardy-Lilttlewood-Sobolev and Stein-Weiss inequalities and integral systems on the Heisenberg group*, Nonl. Anal., **75** (2012), 4296–4314.

[32] Y. HAN, *An integral type Brezis-Nirenberg problem on the Heisenberg group*, J. Diff. Equ., **269** (2020), 4544–4565.

[33] Y. HAN, *Integral equations on compact CR manifold*, Discrete Contin. Dyn. Syst-A, **41** (5) (2021), 2187–2204.

[34] Y. HAN, P. NIU, *Hardy-Sobolev type inequalities on the H-type group*, Manuscripta Math., **118** (2005), 235–252.

[35] Y. HAN, X. WANG, M. ZHU, *Characterization by symmetry of solutions of a nonlinear subelliptic equation on the Heisenberg group*, J. Math. Study, **50** (1) (2017), 17–27.

[36] Y. HAN, S. ZHANG, *Sharp Sobolev inequalities on the complex sphere*, Math. Ineq. Appl., **23** (2020), 149–159.

[37] Y. HAN, M. ZHU, *Hardy-Littlewood-Sobolev inequalities on compact Riemannian manifolds and applications*, J. Diff. Equ., **260** (2016), 1–25.

[38] F. HANG, X. WANG, X. YAN, *An integral equation in conformal geometry*, Ann. Inst. H. Poincaré Analyse Non Linéaire **26** (2009), 1–21.

[39] F. HANG, X. WANG, X. YAN, *Sharp integral inequalities for harmonic functions*, Comm. Pure Appl. Math., **61** (2008), 0054–0095.

[40] G. H. HARDY, J. E. LITTLEWOOD, *Some properties of fractional integrals* (1), Math. Zeitschr. **27** (1928), 565–606.

[41] G. H. HARDY, J. E. LITTLEWOOD, *On certain inequalities connected with the calculus of variantions*, J. London Math. Soc., **5** (1930), 34–39.

[42] D. JERISON, J. M. LEE, *The Yamabe problem on CR manifolds*, J. Diff. Geom. **25** (1987), 167–197.

[43] D. JERISON, J. M. LEE, *Extremals for the Sobolev inequality on the Heisenberg group and the CR Yamabe problem*, J. Amer. Math. Soc., **1** (1988), 1–13.

[44] J. M. LEE, T. H. PARKER, *The Yamabe problem*, Bull. Amer. Math. Soc., **17** (1987), 37–91.

[45] Y. Y. LI, *Remark on some conformally invariant integral equations: the method of moving spheres*, J. Eur. Math. Soc., **6** (2004), 153–180.

[46] E. H. LIEB, *Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities*, Ann. of Math., **118** (1983), 349–374.

[47] E. H. LIEB, M. LOSS, *Analysis*, vol. 14 of Graduate Studies in Mathematics, American Mathemaical Society, Providentce, RI, second edition, 2001.

[48] J. MOSER, *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J. **20** (1970/71), 1077–1092.

[49] Q. A. NGÔ, V. NGUYEN, *Sharp reversed Hardy-Littlewood-Sobolev inequality on $\mathbb{R}^n$*, Israel J. Math., **220** (2017), 189–223.

[50] Q. A. NGÔ, V. NGUYEN, *Sharp reversed Hardy-Littlewood-Sobolev inequality: The case of half space $\mathbb{R}^n_+$*, Int. Math. Res. Not., **20** (2017), 6187–6230.

[51] P. NIU, H. ZHANG, Y. WANG, *Hardy type and Rellich type inequalities on the Heisenberg group*, Proc. Amer. Math. Soc., **129** (2001), 3623–3630.

[52] M. OBATA, *The conjectures on conformal transformations of Riemannian manifolds*, J. Diff. Geom., **6** (1971), 247–258.

[53] S. L. SOBOLEV, *On a theorem of functional analysis*, Mat. Sb. (N.S.) **4** (1938), 471–479, A. M. S. Transl. Ser. 2, **34** (1963), 39–68.

[54] E. M. STEIN, G. WEISS, *Fractional integrals on n-dimensional Euclidean spaces*, J. Math. Mech., **7** (1958), 503–514.

[55] C. TAO, *Reversed Stein-Weiss inequalities with Poisson-type kernel and qualitative analysis of extremal functions*, Adv. Nonlinear Stud., **21** (1) (2021), 167–187.

[56] C. TAO, Y. WANG, *Integral inequalities with an extended Poisson kernel and the existence of the extremals*, Adv. Nonlinear Stud. **23** (2023), 20230104.

[57] N. S. TRUDINGER, *On imbeddings into Orlicz spaces and some applications*, J. Math. Mech., **17** (1967), 473–483.

[58] S. ZHANG AND Y. HAN, *Rearrangement free method for Hardy-Littlewood-Sobolev inequalities on $\mathbb{S}^n$*, Anal. Theory Appl. **38** (2) (2022), 178–203.

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