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ABSTRACT HARDY INEQUALITIES: THE CASE $p = 1$

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Abstract. The Boundedness of an abstract formulation of Hardy operators between Lebesgue spaces over general measure spaces is studied and, when the domain is L^1, shown to be equivalent to the existence of a Hardy inequality on the half line with general Borel measures. This is done by extending the greatest decreasing minorant construction to general measure spaces depending on a totally ordered collection of measurable sets, called an ordered core. A functional description of the greatest decreasing minorant is given, and for a large class of ordered cores, a pointwise description is provided. As an application, characterizations of Hardy inequalities for metric measure spaces are given, we note that the metric measure space is not required to admit a polar decomposition.

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