

## JENSEN TYPE INEQUALITIES FOR ( $m, M, \psi$ )-CONVEX FUNCTIONS WITH APPLICATIONS

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*Abstract.* Among various generalized classes of convexity, the class of  $(m, M, \psi)$ -convex functions, introduced by Dragomir in 2001, has attracted increasing attention recently. This class covers many other subclasses of convexity, such as the class of strongly convex functions, delta convex functions, approximately convex functions and others. In this paper, we present the Jensen and the Jensen-Steffensen type inequalities for  $(m, M, \psi)$ -convex functions. Our results extend and improve the corresponding results valid for different subclasses of convex functions. As application of the main results, we derive new lower and upper bounds estimations for some well-known mean inequalities.

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