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INERTIA INDICES AND THE CONVERSE OF WEYL'S EIGENVALUE INEQUALITY FOR HERMITIAN TENSORS

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Abstract. In this paper, we use inertia indices to present a necessary and sufficient condition in order for eigenvalue inequalities to hold between two Hermitian tensors. As an application, we establish the converse of Weyl's eigenvalue inequality for Hermitian tensors. Also, we prove some classical eigenvalue inequalities for Hermitian tensors. More precisely, we extend Cauchy's interlacing theorem, Weyl's inequality, the monotonicity theorem, and the inclusion principle theorem from matrices to tensors, in a simple and unified approach.

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Paper MIA-28-03

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