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EXTENSIONS OF CLASSICAL ANKENY–RIVLIN INEQUALITY TO THE s^{th} DERIVATIVE

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Abstract. In this paper, we present for a polynomial $p(z)$ of degree n, the s^{th} derivative $(0 \leqslant s < n)$ concept on a result due to Govil et al. [*Illinois J. Math.*, **23** (1979), 319–329]. As an application of this result, we obtain improved generalizations of the well-known theorem due to Ankeny and Rivlin which states that if $p(z)$ is a polynomial of degree n such that $p(z)$ has no zero in $|z| < 1$, then

$$\max_{|z|=R\geqslant 1} |p(z)| \leqslant \left(\frac{R^n+1}{2}\right) \max_{|z|=1} |p(z)|.$$

Moreover, these achievements lead to enhancements of a previous result attributed to Jain [*Turk. J. Math.*, **31** (2007), 89–94] which we have compared by considering a concrete numerical example and analyzed graphically to illustrate their sharpness.

Mathematics subject classification (2020): 30A10, 30C10, 30C15.
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