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NEW INEQUALITIES FOR THE HADAMARD PRODUCT OF HILBERT SPACE OPERATORS

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Abstract. The main goal of this paper is to present further investigations of the Hadamard product of Hilbert space operators and matrices. In particular, we prove a Cauchy-Schwarz-type inequality involving the Hadamard product.

Then, singular value and norm bounds will be obtained as an application of the aforementioned Cauchy-Schwarz inequality. For example, if A and B are compact operators on a separable Hilbert space, it is shown that

$$s_j(A \circ B) \leqslant \||A^*| \circ |B^*|\|^{\frac{1}{2}} s_j^{\frac{1}{2}}(|A| \circ |B|)$$

where $\circ$, $\|\cdot\|$ and $|\cdot|$ denote the Hadamard product, the usual operator norm, and the absolute value, respectively.

After that, numerical radius and spectral radius bounds for operator forms involving the Hadamard product are presented.

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