Mathematical
Inequalities
& Applications
Volume 28, Number 2 (2025), 299–306
doi:10.7153/mia-2025-28-20

CONTINUOUS RANKIN BOUND FOR HILBERT AND BANACH SPACES

K. MAHESH KRISHNA

Abstract. Let (Ω,μ) be a finite measure space and $\{\tau_\alpha\}_{\alpha\in\Omega}$ be a normalized continuous Bessel family for a real Hilbert space $\mathscr{H}$. If the diagonal $\Delta := \{(\alpha,\alpha) : \alpha\in\Omega\}$ is measurable in the measure space $\Omega\times\Omega$, then we show that

$$\sup_{\alpha,\beta\in\Omega,\alpha\neq\beta}\langle\tau_\alpha,\tau_\beta\rangle \geqslant \frac{-(\mu\times\mu)(\Delta)}{(\mu\times\mu)((\Omega\times\Omega)\setminus\Delta)}. \tag{1}$$

We call Inequality (1) as continuous Rankin bound. It improves 77 years old result of Rankin [*Ann. of Math., 1947*]. It also answers one of the questions asked by K. M. Krishna in the paper [Continuous Welch bounds with applications, *Commun. Korean Math. Soc., 2023*]. We also derive Banach space version of Inequality (1).

Mathematics subject classification (2020): 42C15.
Keywords and phrases: Rankin bound, continuous Bessel family, Hilbert space, Banach space.

REFERENCES

[1] S. TWAREQUE ALI, J. P. ANTOINE, AND J. P. GAZEAU, *Continuous frames in Hilbert space*, Ann. Physics, **222**, 1 (1993), 1–37.

[2] SYED TWAREQUE ALI, J. P. ANTOINE, AND J. P. GAZEAU, *Coherent states, wavelets, and their generalizations*, Springer, New York, 2014.

[3] BRUCE C. BERNDT, WINFRIED KOHNEN, AND KEN ONO, *The life and work of R. A. Rankin (1915–2001)*, Ramanujan J., **7** (2003), 9–38.

[4] J. W. S. CASSELS, *An introduction to the geometry of numbers*, Classics in Mathematics, Springer-Verlag, Berlin, 1997.

[5] JOHN H. CONWAY, RONALD H. HARDIN, AND NEIL J. A. SLOANE, *Packing lines, planes, etc.: packings in Grassmannian spaces*, Experiment. Math., **5**, 2 (1996), 139–159.

[6] I. S. DHILLON, R. W. HEATH, JR., T. STROHMER, AND J. A. TROPP, *Constructing packings in Grassmannian manifolds via alternating projection*, Experiment. Math., **17**, 1 (2008), 9–35.

[7] JEAN P. GABARDO AND DEGUANG HAN, *Frames associated with measurable spaces*, Adv. Comput. Math., **18** (2003), 127–147.

[8] THOMAS C. HALES, *Historical overview of the Kepler conjecture*, Discrete Comput. Geom., **36**, 1 (2006), 5–20.

[9] GERALD KAISER, *A friendly guide to wavelets*, Modern Birkhäuser Classics. Birkhäuser/Springer, New York, 2011.

[10] K. MAHESH KRISHNA, *Feichtinger conjectures, R_ε-conjectures and Weaver's conjectures for Banach spaces*, arXiv:2201.00125v1 [math.FA], 1 January, 2022.

[11] K. MAHESH KRISHNA, *Continuous Welch bounds with applications*, Commun. Korean Math. Soc., **38**, 3 (2023), 787–805.

[12] R. A. RANKIN, *On the closest packing of spheres in n dimensions*, Ann. of Math. (2), **48** (1947), 1062–1081.

[13] R. A. RANKIN, *The closest packing of spherical caps in n dimensions*, Proc. Glasgow Math. Assoc., **2** (1955), 139–144.

[14] THOMAS STROHMER AND ROBERT W. HEATH, JR., *Grassmannian frames with applications to coding and communication*, Appl. Comput. Harmon. Anal., **14**, 3 (2003) 257–275.

[15] GEORGE G. SZPIRO, *Kepler's conjecture: How some of the greatest minds in history helped solve one of the oldest math problems in the world*, John Wiley & Sons, Inc., Hoboken, NJ, 2003.
[16] MICHEL TALAGRAND, *Pettis integral and measure theory*, Mem. Amer. Math. Soc., **51**, 307 (1984), ix+224.
[17] JOEL A. TROPP, *Topics in sparse approximation*, (2004) Thesis (Ph.D.), The University of Texas at Austin.
[18] CHUANMING ZONG, *Sphere packings*, Universitext, Springer-Verlag, New York, 1999.