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SOME PROPERTIES OF NORMALIZED REMAINDERS OF THE MACLAURIN EXPANSION FOR A FUNCTION ORIGINATING FROM AN INTEGRAL REPRESENTATION OF THE RECIPROCAL OF THE GAMMA FUNCTION

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Abstract. It is well known that the classical Euler gamma function $\Gamma(z)$ has had very extensive applications in mathematical sciences, including physics and engineering, in the past centuries. In this study, the authors introduce the normalized remainder $T_{2n+1}[\Phi(\theta)]$ of the Maclaurin expansion of the function $\Phi(\theta)=1-\frac{\theta}{\tan\theta}+\ln\frac{\theta}{\sin\theta}$ for $\theta\in(-\pi,\pi)$, which is contained in an integral representation of the reciprocal $\frac{1}{\Gamma(z)}$. In light of the increasing property of two sequences involving the ratio of two non-zero Bernoulli numbers and with the aid of the monotonicity rule for the ratio of two Maclaurin series, they present the logarithmic convexity of the normalized remainder $T_{2n+1}[\Phi(\theta)]$ for $n\in\mathbb{N}_0=\{0,1,2,\dots\}$ in $\theta\in(-\pi,\pi)$ and discuss the monotonicity of the ratio $\frac{T_{2n+3}[\Phi(\theta)]}{T_{2n+1}[\Phi(\theta)]}$ for $n\in\mathbb{N}_0$ in $\theta\in(-\pi,0)\cup(0,\pi)$.

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