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REMARKS ON EXTREMAL FUNCTIONS FOR THE ANISOTROPIC TRUDINGER–MOSER INEQUALITIES INVOLVING L^p NORM

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Abstract. Let $W^{1,n}(\mathbb{R}^n)$ $(n \geqslant 2)$ be the standard Sobolev space, and denote, for $p > n$

$$\gamma_1 = \inf_{u \in W^{1,n}(\mathbb{R}^n),\, u \not\equiv 0} \frac{\int_{\mathbb{R}^n} (F^n(\nabla u) + |u|^n) dx}{(\int_{\mathbb{R}^n} |u|^p dx)^{\frac{n}{p}}},$$

where $F : \mathbb{R}^n \to [0, \infty)$ be a convex function of class $C^2(\mathbb{R}^n \backslash \{0\})$, which is even and positively homogeneous of degree 1. For $\gamma \in [0, \gamma_1)$, we define a norm in $W^{1,n}(\mathbb{R}^n)$ by

$$\|u\|_{F,n,\gamma,p} = \left(\int_{\mathbb{R}^n} (F^n(\nabla u) + |u|^n) dx - \gamma(\int_{\mathbb{R}^n} |u|^p dx)^{\frac{n}{p}} \right)^{\frac{1}{n}}.$$

By performing a blow-up analysis, we prove that for real numbers $0 \leqslant \gamma < \gamma_1$ and $p > n$, the following anisotropic Trudinger-Moser inequality

$$\sup_{u \in W^{1,n}(\mathbb{R}^n), \|u\|_{F,n,\gamma,p} \leqslant 1} \int_{\mathbb{R}^n} \Phi(\lambda_n |u|^{\frac{n}{n-1}}) dx$$

can be attained by some function $u_0 \in W^{1,n}(\mathbb{R}^n)$ with $\|u_0\|_{F,n,\gamma,p} = 1$, where $\Phi(t) = e^t - \sum_{j=0}^{n-1} \frac{t^j}{j!}$, $\lambda_n = n^{\frac{n}{n-1}} \kappa_n^{\frac{1}{n-1}}$ and κ_n is the volume of the unit Wulff ball. In the case $\gamma = 0$, this is reduced to a result of Zhou-Zhou [19].

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Paper MIA-28-30

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