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q–DEFORMED HILBERT TRANSFORM AND ITS RELATED PROPERTIES AND INEQUALITIES

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Abstract. We present a new formulation of the Hilbert transform constructed via the q-deformation of convolution, which is called the q-deformed Hilbert transform. We also find the q-deformed Hilbert transform of some basic functions and examine its connection with the q-Fourier transform. In particular, a number of new related inequalities and embeddings are proved such as a q-analuge of the Chebyshev inequality and a Hardy-type inequality. In additionally, we present a direct application of Hardy-type inequality to study some inequalities for the q-deformed Hilbert transform on $L^p(\mathbb{R}_q)$ and $L^{p,r}(\mathbb{R}_q)$. Finally, we prove a weak (1.1) inequality for the q-deformed Hilbert transform.

Mathematics subject classification (2020): 46A30, 33D15, 47G10, 47B38.

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