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MONOTONIC SEQUENCES AND INEQUALITIES INVOLVING THE RATIO BETWEEN TWO ADJACENT NONZERO BERNOULLI NUMBERS

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Abstract. In the work, the authors concisely review monotonicity results of several sequences involving the ratio between two adjacent nonzero Bernoulli numbers, establish new monotonicity results of several sequences involving the ratio between two adjacent Bernoulli numbers, derive several inequalities for the ratio of two adjacent nonzero Bernoulli numbers, recover the first double inequality for the ratio of two adjacent nonzero Bernoulli numbers, and discover that the function $\left(2^{t-1}-1\right)\zeta(t)$ is logarithmically concave in $t\in(0,\infty)$, where $\zeta(t)$ denoted the Riemann zeta function.

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