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ESTIMATES FOR GENERALIZED FRACTIONAL INTEGRALS ASSOCIATED WITH OPERATORS ON MORREY–CAMPANATO SPACES

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Abstract. Let $\mathscr{L}$ be the infinitesimal generator of an analytic semigroup $\{e^{-t\mathscr{L}}\}_{t>0}$ satisfying the Gaussian upper bounds. For given $0<\alpha<n$, let $\mathscr{L}^{-\alpha/2}$ be the generalized fractional integral associated with $\mathscr{L}$, which is defined as

$$\mathscr{L}^{-\alpha/2}(f)(x) := \frac{1}{\Gamma(\alpha/2)}\int_0^{+\infty} e^{-t\mathscr{L}}(f)(x)t^{\alpha/2-1}\,dt,$$

where $\Gamma(\cdot)$ is the usual gamma function. For a locally integrable function $b(x)$ defined on $\mathbb{R}^n$, the related commutator operator $\big[b,\mathscr{L}^{-\alpha/2}\big]$ generated by b and $\mathscr{L}^{-\alpha/2}$ is defined by

$$\big[b,\mathscr{L}^{-\alpha/2}\big](f)(x) := b(x)\cdot\mathscr{L}^{-\alpha/2}(f)(x) - \mathscr{L}^{-\alpha/2}(bf)(x).$$

A new class of Morrey–Campanato spaces associated with $\mathscr{L}$ is introduced in this paper. The authors establish some new estimates for the commutators $\big[b,\mathscr{L}^{-\alpha/2}\big]$ on Morrey–Campanato spaces. The corresponding results for higher-order commutators $\big[b,\mathscr{L}^{-\alpha/2}\big]^m$ $(m\in\mathbb{N})$ are also discussed.

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REFERENCES

[1] D. R. ADAMS, *A note on Riesz potentials*, Duke Math. J., **42** (1975), 765–778.
[2] D. R. ADAMS, *Morrey Spaces*, Lecture notes in applied and numerical harmonic analysis, Birkhäuser/Springer, Cham, 2015.
[3] P. AUSCHER AND J. M. MARTELL, *Weighted norm inequalities for fractional operators*, Indiana Univ. Math. J., **57** (2008), 1845–1870.
[4] A. BENYI, J. M. MARTELL, K. MOEN, E. STACHURA AND R. H. TORRES, *Boundedness results for commutators with BMO functions via weighted estimates: a comprehensive approach*, Math. Ann. **376** (2020), 61–102.
[5] S. CAMPANATO, *Proprietàdi Hölderianità di alcune classi di funzioni*, Ann. Scuola Norm. Sup. Pisa., **17** (1963), 173–188.
[6] S. CHANILLO, *A note on commutators*, Indiana Univ. Math. J., **31** (1982), 7–16.
[7] D. CRUZ-URIBE, J. M. MARTELL AND C. PÉREZ, *Extrapolation from A_∞ weights and applications*, J. Funct. Anal, **213** (2004), 412–439.
[8] D. G. DENG, X. T. DUONG AND L. X. YAN, *A characterization of the Morrey–Campanato spaces*, Math. Z., **250** (2005), 641–655.
[9] D. G. DENG, X. T. DUONG, A. SIKORA AND L. X. YAN, *Comparison of the classical BMO with the BMO spaces associated with operators and applications*, Rev. Mat. Iberoamericana, **24** (2008), 267–296.
[10] X. T. DUONG AND L. X. YAN, *On commutators of fractional integrals*, Proc. Amer. Math. Soc., **132** (2004), 3549–3557.

Paper MIA-29-02

[11] X. T. DUONG AND L. X. YAN, *New function spaces of BMO type, the John–Nirenberg inequality, interpolation and applications*, Comm. Pure Appl. Math., **58** (2005), 1375–1420.
[12] X. T. DUONG AND L. X. YAN, *Duality of Hardy and BMO spaces associated with operators with heat kernel bounds*, J. Amer. Math. Soc., **18** (2005), 943–973.
[13] L. GRAFAKOS, *Modern Fourier Analysis*, Third Edition, Springer-Verlag, 2014.
[14] S. JANSON, M. H. TAIBLESON AND G. WEISS, *Elementary characterizations of the Morrey–Campanato spaces*, Lecture Notes in Math., **992** (1983), 101–114.
[15] F. JOHN AND L. NIRENBERG, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math., **14** (1961), 415–426.
[16] Y. KOMORI AND S. SHIRAI, *Weighted Morrey spaces and a singular integral operator*, Math. Nachr., **282** (2009), 219–231.
[17] S. Z. LU, Y. DING AND D. Y. YAN, *Singular Integrals and Related Topics*, World Scientific Publishing, NJ, 2007.
[18] J. M. MARTELL, *Sharp maximal functions associated with approximations of the identity in spaces of homogeneous type and applications*, Studia Math., **161** (2004), 113–145.
[19] H. X. MO AND S. Z. LU, *Boundedness of multilinear commutators of generalized fractional integrals*, Math. Nachr., **281** (2008), 1328–1340.
[20] C. B. MORREY, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc., **43** (1938), 126–166.
[21] B. MUCKENHOUPT AND R. L. WHEEDEN, *Weighted norm inequalities for singular and fractional integrals*, Trans. Amer. Math. Soc., **161** (1971), 249–258.
[22] B. MUCKENHOUPT AND R. L. WHEEDEN, *Weighted norm inequalities for fractional integrals*, Trans. Amer. Math. Soc., **192** (1974), 261–274.
[23] J. PEETRE, *On the theory of $\mathscr{L}_{p,\lambda}$ spaces*, J. Funct. Anal., **4** (1969), 71–87.
[24] M. PALUSZYŃSKI, *Characterization of the Besov spaces via the commutator operator of Coifman, Rochberg and Weiss*, Indiana Univ. Math. J., **44** (1995), 1–17.
[25] C. SEGOVIA AND J. L. TORREA, *Weighted inequalities for commutators of fractional and singular integrals*, Publ. Mat., **35** (1991), 209–235.
[26] S. G. SHI AND S. Z. LU, *Some characterizations of Campanato spaces via commutators on Morrey spaces*, Pacific J. Math., **264** (2013), 221–234.
[27] S. G. SHI AND S. Z. LU, *A characterization of Campanato space via commutator of fractional integral*, J. Math. Anal. Appl., **419** (2014), 123–137.
[28] E. M. STEIN, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, New Jersey, 1970.
[29] H. WANG, *Boundedness of fractional integral operators with rough kernels on weighted Morrey spaces*, Acta Math. Sinica (Chin. Ser.), **56** (2013), 175–186.
[30] H. WANG, *Some estimates for commutators of fractional integrals associated to operators with Gaussian kernel bounds on weighted Morrey spaces*, Anal. Theory Appl., **29** (2013), 72–85.
[31] H. WANG, *Estimates for fractional integral operators and linear commutators on certain weighted amalgam spaces*, J. Funct. Spaces 2020, Art. ID 2697104, 25 pp.