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NEW ASYMPTOTICS AND INEQUALITIES RELATED TO THE VOLUME OF THE UNIT BALL IN $\mathbb{R}^n$

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Abstract. Let $\Omega_n = \pi^{n/2}/\Gamma(\frac{n}{2}+1)\,(n \in \mathbb{N})$ denote the volume of the unit ball in $\mathbb{R}^n$. Define the function $I(x)$ by

$$I(x) = \frac{\Omega_{2x}^2}{\Omega_{2x-1}\Omega_{2x+1}} = \left(x+\frac{1}{2}\right)\left[\frac{\Gamma(x+\frac{1}{2})}{\Gamma(x+1)}\right]^2,$$

where $\Omega_x = \pi^{x/2}/\Gamma(\frac{x}{2}+1)$. In this paper, we present asymptotic expansions of the function $I(x)$, and then establish asymptotic expansions and inequalities of the quantity $\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}$. Also, we prove that the function $F(x) = (1+\frac{1}{x})^{1/4}/I(x)$ is logarithmically completely monotonic on $(0,\infty)$, which derives a double inequality for the quantity $\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}$.

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