Mathematical
Inequalities
& Applications
Volume 29, Number 2 (2026), 137–155
doi:10.7153/mia-2026-29-08

PATULA, HARTMAN—WINTNER AND LYAPUNOV TYPE INEQUALITIES FOR A CLASS OF STURM—LIOUVILLE PROBLEMS

MOHAMED JLELI AND BESSEM SAMET*

Abstract. We derive Patula, Hartman–Wintner, and Lyapunov type inequalities for Sturm–Liouville problems of the form

$$-u''(x)+g(x)u(x)=f(x)u(x), \qquad a<x<b,$$

where $f,g\in C([a,b])$ and $g\geqslant 0$. We then apply these inequalities to several special cases. In particular, we obtain a refinement of Bargmann's inequality for the radial Schrödinger equation.

Mathematics subject classification (2020): 34B05, 34C10, 26D10.

Keywords and phrases: Sturm-Liouville problems, Patula type inequalities, Hartman-Wintner type inequalities, Lyapunov type inequalities, radial Schrödinger equation, Airy differential equation.

REFERENCES

[1] R. P. AGARWAL, M. BOHNER AND A. ÖZBEKLER, *Lyapunov Inequalities and Applications*, Springer, Berlin (2021).

[2] R. P. AGARWAL, M. JLELI AND B. SAMET, *On De La Vallée Poussin-type inequalities in higher dimension and applications*, Appl. Math. Lett., **86**, (2018), 264–269.

[3] M. F. AKTAŞ, *On Lyapunov-type inequalities for nonlinear Hamiltonian-type problems*, Filomat, **36**, 10 (2022), 3423–3432.

[4] M. F. AKTAŞ, D. ÇAKMAK AND A. AHMETOĞLU, *Lyapunov-type inequalities for fourth-order boundary value problems*, Rev. Real Acad. Cienc. Exactas Fís. Nat. Ser. A Mat., **113**, 2 (2019), 615–625.

[5] M. F. AKTAŞ AND B. B. ERÇIKTI, *On Lyapunov-type inequalities for five different types of higher order boundary value problems*, Turk. J. Math., **48**, 1 (2024), 90–105.

[6] V. BARGMANN, *On the number of bound states in a central field of force*, Proc. Nat. Acad. Sci. USA, **38**, 11 (1952), 961–966.

[7] R. C. BROWN AND D. B. HINTON, *Lyapunov inequalities and their applications*, in: *Survey on Classical Inequalities* (Ed. T. M. Rassias), Math. Appl. **517**, Kluwer Acad. Publ., Dordrecht–London (2000), 1–25.

[8] K. M. DAS AND A. S. VATSALA, *Green's function for $n-n$ boundary value problem and an analogue of Hartman's result*, J. Math. Anal. Appl., **51**, 3 (1975), 670–677.

[9] S. DHAR AND J. S. KELLY, *Lower bounds for eigenvalues of even ordered quasilinear differential equations*, Proc. Amer. Math. Soc., **151**, 2 (2023), 647–661.

[10] R. A. C. FERREIRA, *On a Lyapunov-type inequality and the zeros of a certain Mittag-Leffler function*, J. Math. Anal. Appl., **412**, 2 (2014), 1058–1063.

[11] Y. HAO AND H. LIU, *Some new Lyapunov-type inequalities for a class of $(n+1)$ th order nonlinear differential equations*, Math. Found. Comput., **8**, 5 (2025), 701–709.

[12] P. HARTMAN AND A. WINTNER, *On an oscillation criterion of Liapunoff*, Am. J. Math., **73**, 4 (1951), 885–890.

[13] M. HASHIZUME AND F. TAKAHASHI, *Lyapunov inequality for an elliptic problem with the Robin boundary condition*, Nonlinear Anal., **129**, (2015), 189–197.

[14] A. KASSYMOV AND B. T. TOREBEK, *Lyapunov-type inequalities for a nonlinear fractional boundary value problem*, Rev. Real Acad. Cienc. Exactas Fís. Nat. Ser. A Mat., **115**, 1 (2021), 1–10.

Paper MIA-29-08

[15] A. LYAPUNOV, *Problème général de la stabilité du mouvement*, Ann. Fac. Sci. Toulouse, **2**, 9 (1907), 204–474.
[16] P. OZA, *Lyapunov-type inequality to general second-order elliptic equations*, Math. Meth. Appl. Sci., **47**, 18 (2024), 14688–14698.
[17] W. T. PATULA, *On the distance between zeroes*, Proc. Amer. Math. Soc., **52**, 1 (1975), 247–251.
[18] B. SAMET AND H. AYDI, *Lyapunov-type inequalities for an anti-periodic fractional boundary value problem involving ψ-Caputo fractional derivative*, J. Inequal. Appl., **2018**, 286 (2018).
[19] J. SÁNCHEZ AND V. VERGARA, *A Lyapunov-type inequality for a ψ-Laplacian operator*, Nonlinear Anal., **74**, 18 (2011), 7071–7077.
[20] O. VALLÉE AND M. SOARES, *Airy Functions and Applications to Physics*, Imperial College Press, London (2004).
[21] A. WINTNER, *On the non-existence of conjugate points*, Am. J. Math., **73**, 2 (1951), 368–380.
[22] R. YANG, I. SIM AND Y. H. LEE, *Lyapunov-type inequalities for one-dimensional Minkowski-curvature problems*, Appl. Math. Lett., **91**, (2019), 188–193.
[23] B. F. ZOHRA, H. BENAOUDA AND M. KIRANE, *Lyapunov- and Hartman-Wintner-type inequalities for a nonlinear fractional BVP with generalized ψ-Hilfer derivative*, Math. Meth. Appl. Sci., **44**, 3 (2021), 2637–2649.