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THREE WEIGHTS HARDY–TYPE INEQUALITIES

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Abstract. In this paper we derive some new weighted iterated discrete Hardy-type inequalities with three independent weight sequences and various parameter ranges. In a special case, the operator reduces to the classical Hardy-type operator. Although integral analogues of such inequalities have been widely studied, the discrete setting has only recently attracted significant attention, including extensions to classes of matrix operators. The aim of the paper is to establish necessary and sufficient conditions for the validity of these inequalities in cases where the parameter p does not exceed one. The obtained results complement known criteria, extend them to new parameter ranges, and provide new insights into borderline cases. Some applications of Hardy-type inequalities are pointed out.

Mathematics subject classification (2020): 26D10, 26D15, 26D20, 47B38.
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