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INTEGRAL REPRESENTATION AND FUNCTIONAL INEQUALITIES INVOLVING GENERALIZED POLYLOGARITHM

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Abstract. In this manuscript, integral representation of generalized polylogarithm is obtained through two different methods. The first one involves the Dirichlet series and its Laplace representation resulting in a single integral representation, related to the Lerch transcendent function. The second approach utilizes the Hadamard convolution, which leads to a double integral representation. As consequences, complete monotonicity, Turán inequality, convexity, and bounds of the generalized polylogarithm are deduced. Finally, an alternative proof of an existing integral representation of the generalized polylogarithm using the Hadamard convolution is provided.

Mathematics subject classification (2020): 33B30, 26D07, 30E20.
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