

BEREZIN NUMBER AND BEREZIN NORM INEQUALITIES: NEW AND IMPROVED RESULTS FOR $n \times n$ OPERATOR MATRICES

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Abstract. Let $T = [T_{jk}]$ be an $n \times n$ operator matrix with $T_{jk} \in \mathcal{CR}(\mathbb{H}_k, \mathbb{H}_j)$, for all $j, k = 1, 2, \dots, n$. In this article, we obtain new and improved upper bounds for Berezin number and Berezin norm of $n \times n$ operator matrices. In particular, we obtain a bound for 2×2 operator matrices,

$$\text{ber} \left(\begin{bmatrix} 0 & T_{12} \\ T_{21} & 0 \end{bmatrix} \right) \leq \frac{1}{2} \left(\min \left\{ \| |T_{12}|^2 + |T_{21}^*|^2 \|_{\text{ber}}^{\frac{1}{2}} \| T_{12} T_{12}^\dagger + T_{21}^\dagger T_{21} \|_{\text{ber}}^{\frac{1}{2}}, \right. \right. \\ \left. \left. \| |T_{12}^*|^2 + |T_{21}|^2 \|_{\text{ber}}^{\frac{1}{2}} \| T_{12}^\dagger T_{12} + T_{21} T_{21}^\dagger \|_{\text{ber}}^{\frac{1}{2}} \right\} \right),$$

which improves the existing bound. Among other estimates, we obtain strengthened Berezin number estimates for $n \times n$ operator matrices and as an application, a Berezin number bound for closed range operators, namely,

$$\text{ber}(T) \leq \left(\frac{1}{4} \| |T^*|^4 + |T|^4 \|_{\text{ber}} + \frac{1}{2} \text{ber}(|T^*|^2 |T|^2) \right)^{\frac{1}{4}} \left(\frac{1}{4} \| T T^\dagger + T^\dagger T \|_{\text{ber}} + \frac{1}{2} \text{ber}(T^\dagger T^2 T^\dagger) \right)^{\frac{1}{4}},$$

which refines the existing bound. Also, we obtain new upper bound for Berezin norm of 2×2 operator matrices and provide an example to show that it is sharper than the existing bound.

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