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TWO LOGARITHMICALLY COMPLETELY MONOTONIC FUNCTIONS INVOLVING LOGARITHMIC FUNCTION

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Abstract. In the paper, using the integral representation of the Bernoulli numbers of the second kind, the authors establish the logarithmically complete monotonicity of two functions involving the logarithmic function, and show that the completely monotonic degree of another derived completely monotonic function involving the logarithmic function is equal to 1. These results resolve previously posed conjectures.

Mathematics subject classification (2020): Primary 26A48; Secondary 26A09, 26A51, 44A10.

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REFERENCES

[1] R. D. ATANASSOV AND U. V. TSOUKROVSKI, *Some properties of a class of logarithmically completely monotonic functions*, C. R. Acad. Bulgare Sci. **41** (1988), no. 2, 21–23.

[2] C. BERG, *Integral representation of some functions related to the gamma function*, Mediterr. J. Math. **1** (2004), no. 4, 433–439, https://doi.org/10.1007/s00009-004-0022-6.

[3] L. COMTET, *Advanced Combinatorics: The Art of Finite and Infinite Expansions*, Revised and Enlarged Edition, D. Reidel Publishing Co., 1974, https://doi.org/10.1007/978-94-010-2196-8.

[4] H. T. DAVIS, *The approximation of logarithmic numbers*, Amer. Math. Monthly **64** (1957), no. 8, part II, 11–18, https://doi.org/10.2307/2308510.

[5] B.-N. GUO AND F. QI, *A completely monotonic function involving the tri-gamma function and with degree one*, Appl. Math. Comput. **218** (2012), no. 19, 9890–9897, https://doi.org/10.1016/j.amc.2012.03.075.

[6] B.-N. GUO AND F. QI, *A property of logarithmically absolutely monotonic functions and the logarithmically complete monotonicity of a power-exponential function*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. **72** (2010), no. 2, 21–30.

[7] X.-L. LIU, T. XU, C.-Y. HE, AND F. QI, *Sharp bounds for even-indexed Bernoulli numbers and their ratios*, Filomat **40** (2026), accepted on 14 June 2026, https://www.researchgate.net/publication/407043457.

[8] D. S. MITRINOVIĆ, J. E. PEČARIĆ, AND A. M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht-Boston-London, 1993, https://doi.org/10.1007/978-94-017-1043-5.

[9] F. QI, *Completely monotonic degree of a function involving trigamma and tetragamma functions*, AIMS Math. **5** (2020), no. 4, 3391–3407, https://doi.org/10.3934/math.2020219.

[10] F. QI, *Properties of modified Bessel functions and completely monotonic degrees of differences between exponential and trigamma functions*, Math. Inequal. Appl. **18** (2015), no. 2, 493–518, https://doi.org/10.7153/mia-18-37.

[11] F. QI, *Uniform treatments of Bernoulli numbers, Stirling numbers, and their generating functions*, arXiv:2504.16965, https://doi.org/10.48550/arXiv.2504.16965.

[12] F. QI AND C.-P. CHEN, *A complete monotonicity property of the gamma function*, J. Math. Anal. Appl. **296** (2004), 603–607, https://doi.org/10.1016/j.jmaa.2004.04.026.

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[13] F. QI, B.-N. GUO, AND C.-P. CHEN, *Some completely monotonic functions involving the gamma and polygamma functions*, J. Aust. Math. Soc. **80** (2006), 81–88, https://doi.org/10.1017/S1446788700011393.

[14] F. QI AND W.-H. LI, *A logarithmically completely monotonic function involving the ratio of gamma functions*, J. Appl. Anal. Comput. **5** (2015), no. 4, 626–634, https://doi.org/10.11948/2015049.

[15] F. QI AND X.-J. ZHANG, *An integral representation, some inequalities, and complete monotonicity of the Bernoulli numbers of the second kind*, Bull. Korean Math. Soc. **52** (2015), no. 3, 987–998, https://doi.org/10.4134/BKMS.2015.52.3.987.

[16] R. L. SCHILLING, R. SONG, AND Z. VONDRAČEK, *Bernstein Functions – Theory and Applications*, 2nd ed., de Gruyter Studies in mathematics **37**, Walter de Gruyter, Berlin, Germany, 2012, https://doi.org/10.1515/9783110269338.

[17] D. V. WIDDER, *The Laplace Transform*, Princeton University Press, 1946.