Mathematical
Inequalities
& Applications
Volume 29, Number 4 (2026), 445–460
doi:10.7153/mia-2026-29-28

BLOCK DECOMPOSITIONS AND SHARP MATRIX PRODUCT BOUNDS

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Abstract. In this paper, we establish new numerical radius, spectral norm, and singular value inequalities for matrix expressions involving matrix products and sums. In particular, we refine the matrix Cauchy-Schwarz inequality by showing that if A,B,X are complex square matrices, with X positive semidefinite, then

$$\|AXB^*\|^2 \leqslant \frac{1}{4}\big(\omega(XA^*A)+\|XA^*A\|\big)\big(\omega(XB^*B)+\|XB^*B\|\big),$$

where $\|\cdot\|$ is the spectral norm and $\omega(\cdot)$ is the numerical radius.

Mathematics subject classification (2020): Primary 15A60; Secondary 47A12, 47A30, 47B10.
Keywords and phrases: Positive semidefinite matrix, numerical radius, spectral norm, block matrix.

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