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INDEFINITE STURM–LIOUVILLE OPERATORS $(\operatorname{sgn} x)\left(-\frac{d^2}{dx^2}+q(x)\right)$ WITH FINITE–ZONE POTENTIALS

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Abstract. The indefinite Sturm-Liouville operator $A = (\operatorname{sgn} x)(-d^2/dx^2 + q)$ is studied. It is proved that similarity of A to a selfadjoint operator is equivalent to integral estimates of Cauchy type integrals. Some simple sufficient and necessary conditions for the similarity to a selfadjoint operator in terms of Weyl functions are given. For operators with a finite-zone potential q, the components $A_{\rm ess}$ and $A_{\rm disc}$ of A corresponding to the essential and the discrete spectrums, respectively, are investigated. The main result of the paper is a criterion of similarity of the operator A (resp. $A_{\rm ess}$) with a finite-zone potential q to a normal (resp. selfadjoint) operator. It is given in terms of the Weyl functions corresponding to the Sturm-Liouville operator $-d^2/dx^2 + q$. Jordan structure of the operator $A_{\rm disc}$ is described. An example of a non-definitizable operator A that is similar to a normal operator is presented too.

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