Operators
and
Matrices
Volume 1, Number 3 (2007), 369–383

HADAMARD DUALS, RETRACTABILITY AND OPPENHEIM'S INEQUALITY

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Abstract. Oppenheim's determinantal inequality was originally proved for positive semidefinite matrices and has produced many interesting consequences and applications. Positive semidefinite matrices were a natural class to consider partly because they are closed under Hadamard (or entry-wise) multiplication. Since Oppenheim's original contribution, others have considered similar inequalities for M-matrices, inverse M-matrices and totally nonnegative matrices. We attempt to unify many of these existing results dealing with Oppenheim's inequality, and our approach relies on two major themes: retractions and Hadamard duals. Retractions are a type of diagonal perturbation and the Hadamard dual is a maximal collection of matrices with a closure property under Hadamard multiplication. These notions are applied to yield results that generalize Oppenheim's original result.

Mathematics subject classification (2000): 15A48, 15A57.
Key words and phrases: Hadamard products, P-matrices, positive semidefinite matrices, M-matrices, inverse M-matrices, Oppenheim's inequality, totally nonnegative matrices, retractability, determinantal inequalities.

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