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WEYL–TITCHMARSH THEORY AND BORG–MARCHENKO–TYPE UNIQUENESS RESULTS FOR CMV OPERATORS WITH MATRIX–VALUED VERBLUNSKY COEFFICIENTS

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Abstract. We prove local and global versions of Borg–Marchenko-type uniqueness theorems for half-lattice and full-lattice CMV operators (CMV for Cantero, Moral, and Velázquez [19]) with matrix-valued Verblunsky coefficients. While our half-lattice results are formulated in terms of matrix-valued Weyl–Titchmarsh functions, our full-lattice results involve the diagonal and main off-diagonal Green's matrices.

We also develop the basics of Weyl–Titchmarsh theory for CMV operators with matrix-valued Verblunsky coefficients as this is of independent interest and an essential ingredient in proving the corresponding Borg–Marchenko-type uniqueness theorems.

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