

## INERTIA THEOREMS BASED ON OPERATOR LYAPUNOV EQUATIONS

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*Abstract.* The well-known Carlson–Schneider inertia theorem for finite matrices, satisfying the Lyapunov equation with a semi-definite right-hand side, is extended to linear operators acting on an infinite dimensional Hilbert space. The proofs use extensively the theory of linear operators acting on indefinite inner product spaces. An application to stability problems of semigroups is also presented.

*Mathematics subject classification (2000):* 47A62, 47B06, 47B50, 47D60, 47N70, 93A99, 15A24.

*Key words and phrases:* Spectrum localization, Lyapunov equation, inertia theorems, bisemigroup.

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