

INVERTIBILITY AND FREDHOLMNESS OF LINEAR COMBINATIONS OF QUADRATIC, k -POTENT AND NILPOTENT OPERATORS

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Abstract. Recently, the invertibility of linear combinations of two idempotents has been studied by several authors. Let P and Q be idempotents in a Banach algebra. It was shown that the invertibility of $P + Q$ is equivalent to that of $aP + bQ$ for nonzero a, b with $a + b \neq 0$. In this note, we obtain a similar result for square zero operators and those operators satisfying $x^2 = dx$ for some scalar d . More generally, we show that if P, Q satisfy a quadratic polynomial $(x - c)(x - d)$ then the linear combination $aP + bQ - c(a + b)$ being invertible or Fredholm (and the index) is independent of the choice of the nonzero scalars a, b . However, this is not the case when P and Q are involutions, unitaries, partial isometries, k -potents ($k \geq 3$) and other nilpotents, as counterexamples are provided.

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