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SELF–ADJOINT EXTENSIONS OF RESTRICTIONS

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Abstract. We provide a simple recipe for obtaining all self-adjoint extensions, together with their resolvent, of the symmetric operator S obtained by restricting the self-adjoint operator $A : \mathscr{D}(A) \subseteq \mathscr{H} \to \mathscr{H}$ to the dense, closed with respect to the graph norm, subspace $\mathscr{N} \subset \mathscr{D}(A)$. Neither the knowledge of S^* nor of the deficiency spaces of S is required. Typically A is a differential operator and $\mathscr{N}$ is the kernel of some trace (restriction) operator along a null subset. We parametrise the extensions by the bundle $p : \mathsf{E}(\mathfrak{h}) \to \mathsf{P}(\mathfrak{h})$, where $\mathsf{P}(\mathfrak{h})$ denotes the set of orthogonal projections in the Hilbert space $\mathfrak{h} \simeq \mathscr{D}(A)/\mathscr{N}$ and $p^{-1}(\Pi)$ is the set of self-adjoint operators in the range of Π. The set of self-adjoint operators in $\mathfrak{h}$, i.e. $p^{-1}(\mathsf{1})$, parametrises the relatively prime extensions. Any $(\Pi, \Theta) \in \mathsf{E}(\mathfrak{h})$ determines a boundary condition in the domain of the corresponding extension $A_{\Pi,\Theta}$ and explicitly appears in the formula for the resolvent $(-A_{\Pi,\Theta} + z)^{-1}$. The connection with both von Neumann's and Boundary Triples theories of self-adjoint extensions is explained. Some examples related to quantum graphs, to Schrödinger operators with point interactions and to elliptic boundary value problems are given.

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