

ASYMPTOTIC PSEUDOMODES OF TOEPLITZ MATRICES

ALBRECHT BÖTTCHER, SERGEI GRUDSKY¹ AND JÉRÉMIE UNTERBERGER

Abstract. Questions in probability and statistical physics lead to the problem of finding the eigenvectors associated with the extreme eigenvalues of Toeplitz matrices generated by Fisher-Hartwig symbols. We here simplify the problem and consider pseudomodes instead of eigenvectors. This replacement allows us to treat fairly general symbols, which are far beyond Fisher-Hartwig symbols. Our main result delivers a variety of concrete unit vectors x_n such that if $T_n(a)$ is the $n \times n$ truncation of the infinite Toeplitz matrix generated by a function $a \in L^1$ satisfying mild additional conditions and λ is in the range of this function, then $\|T_n(a)x_n - \lambda x_n\| \rightarrow 0$.

Mathematics subject classification (2000): 47B35, 15A18, 41A80, 46N30.

Keywords and phrases: Toeplitz matrix, Fisher-Hartwig symbol, eigenvector, pseudomode, fractional Brownian motion.

REFERENCES

- [1] M. N. BARBER AND M. E. FISHER, *Critical Phenomena in Systems of Finite Thickness, I. The Spherical Model*, Annals of Physics **77** (1973), 1–78.
- [2] J. G. BRANKOV, D. M. DANCHEV, AND N. S. TONCHEV, *Theory of critical phenomena in finite-size systems*, Series in Modern Condensed Matter Physics Vol. 9, World Scientific, Singapore (2000).
- [3] T. H. BERLIN AND M. KAC, *The spherical model of a ferromagnet*, Phys. Rev. **86** (1952), 821–835.
- [4] A. BÖTTCHER, M. EMBREE, AND L. N. TREFETHEN, *Piecewise continuous Toeplitz matrices and operators: slow approach to infinity*, SIAM J. Matrix Analysis Appl. **24** (2002), 484–489.
- [5] A. BÖTTCHER AND S. GRUDSKY, *Toeplitz matrices with slowly growing pseudospectra*, In: Factorization, Singular Operators and Related Problems in Honour of Georgii Litvinchuk (S. Samko, A. Lebre, A. F. dos Santos, eds.), pp. 43–54, Kluwer Academic Publishers, Dordrecht 2003.
- [6] A. BÖTTCHER AND S. GRUDSKY, *Asymptotically good pseudomodes for Toeplitz matrices and Wiener-Hopf operators*, Oper. Theory Adv. Appl. **147** (2004), 175–188.
- [7] A. BÖTTCHER AND B. SILBERMANN, *Toeplitz matrices and determinants with Fisher-Hartwig symbols*, J. Funct. Anal. **63** (1985), 178–214.
- [8] A. BÖTTCHER AND J. VIRTANEN, *Norms of Toeplitz matrices with Fisher-Hartwig symbols*, SIAM J. Matrix Analysis Appl. **29** (2007), 660–671.
- [9] I. GOHBERG AND I. A. FELDMAN, *Convolution Equations and Projection Methods for Their Solution*, Amer. Math. Soc., Providence 1974.
- [10] M. KAC, W. L. MURDOCK, AND G. SZEGÖ, *On the eigenvalues of certain Hermitian forms*, J. Rational Mech. Anal. **2** (1953), 767–800.
- [11] S. V. PARTER, *Extreme eigenvalues of Toeplitz forms and applications to elliptic difference equations*, Trans. Amer. Math. Soc. **99** (1961), 153–192.
- [12] S. V. PARTER, *On the extreme eigenvalues of Toeplitz matrices*, Trans. Amer. Math. Soc. **100** (1961), 263–276.
- [13] L. REICHEL AND L. N. TREFETHEN, *Eigenvalues and pseudo-eigenvalues of Toeplitz matrices*, Linear Algebra Appl. **162/164** (1992), 153–185.
- [14] G. SAMORODNITSKY AND M. TAQQU, *Stable Non-Gaussian Random Processes*, Chapman and Hall, New York (1994).
- [15] S. SERRA, *On the extreme spectral properties of Toeplitz matrices generated by L^1 functions with several minima/maxima*, BIT **36** (1996), 135–142.

- [16] S. SERRA, *On the extreme eigenvalues of Hermitian (block) Toeplitz matrices*, Linear Algebra Appl. **270** (1998), 109–129.
- [17] L. N. TREFETHEN AND M. EMBREE, *Spectra and Pseudospectra, The Behavior of Nonnormal Matrices and Operators*, Princeton University Press, Princeton 2005.
- [18] H. WIDOM, *On the eigenvalues of certain Hermitian operators*, Trans. Amer. Math. Soc. **88** (1958), 491–522.
- [19] N. L. ZAMARASHKIN AND E. E. TYRTYSHNIKOV, *Distribution of the eigenvalues and singular numbers of Toeplitz matrices under weakened requirements on the generating function*, Sb. Math. **188** (1997), 1191–1201.
- [20] A. ZYGMUND, *Trigonometric Series*, Vol. I, University of Cambridge Press, Cambridge 1959.