

GENERATORS OF II_1 FACTORS

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Abstract. In 2005, Junhao Shen introduced a new invariant, $\mathcal{G}(N)$, of a diffuse von Neumann algebra N with a fixed faithful trace, and he used this invariant to give a unified approach to showing that large classes of II_1 factors M are singly generated. This paper focuses on properties of this invariant. We relate $\mathcal{G}(M)$ to the number of self-adjoint generators of a II_1 factor M : if $\mathcal{G}(M) < n/2$, then M is generated by $n + 1$ self-adjoint operators, whereas if M is generated by $n + 1$ self-adjoint operators, then $\mathcal{G}(M) \leq n/2$. The invariant $\mathcal{G}(\cdot)$ is well-behaved under amplification, satisfying $\mathcal{G}(M_t) = t^{-2}\mathcal{G}(M)$ for all $t > 0$. In particular, if $\mathcal{G}(\mathcal{L}\mathbb{F}_r) > 0$ for any particular $r > 1$, then the free group factors are pairwise non-isomorphic and are not singly generated for sufficiently large values of r . Estimates are given for forming free products and passing to finite index subfactors and the basic construction. We also examine a version of the invariant $\mathcal{G}_{\text{sa}}(M)$ defined only using self-adjoint operators; this is proved to satisfy $\mathcal{G}_{\text{sa}}(M) = 2\mathcal{G}(M)$. Finally we give inequalities relating a quantity involved in the calculation of $\mathcal{G}(M)$ to the free-entropy dimension δ_0 of a collection of generators for M .

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REFERENCES

- [1] R. G. DOUGLAS AND C. PEARCY, *Von Neumann algebras with a single generator*, Michigan Math. J., 16:21–26, 1969.
- [2] K. DYKEMA AND U. HAAGERUP, *Invariant subspaces of the quasinilpotent DT-operator*, J. Funct. Anal., 209(2):332–366, 2004.
- [3] K. J. DYKEMA, *Free products of hyperfinite von neumann algebras and free dimension*, Duke Math. J., 69(1):97–119, 1993.
- [4] K. J. DYKEMA, *Interpolated free group factors*, Pacific J. Math., 163(1):123–135, 1994.
- [5] K. J. DYKEMA AND U. HAAGERUP, *DT-operators and decomposability of Voiculescu’s circular operator*, Amer. J. Math., 126(1):121–189, 2004.
- [6] K. J. DYKEMA, K. JUNG, AND D. SHLYAKHTENKO, *The microstates free entropy dimension of any DT-operator is 2*, Doc. Math., 10:247–261 (electronic), 2005.
- [7] L. GE, *Applications of free entropy to finite von neumann algebras*, Amer. J. Math., 119(2):467–485, 1997.
- [8] L. GE AND S. POPA, *On some decomposition properties for factors of type II_1* , Duke Math. J., 94(1):79–101, 1998.
- [9] L. GE AND J. SHEN, *Generator problem for certain property T factors*, Proc. Natl. Acad. Sci. USA, 99(2):565–567 (electronic), 2002.
- [10] D. HADWIN AND J. SHEN, *Free orbit dimension of finite von Neumann algebras*, J. Funct. Anal., 249(1):75–91, 2007.
- [11] V. F. R. JONES, *Index for subfactors*, Invent. Math., 72(1):1–25, 1983.
- [12] K. JUNG, *A free entropy dimension lemma*, Pacific J. Math., 211(2):265–271, 2003.
- [13] K. JUNG, *The free entropy dimension of hyperfinite von Neumann algebras*, Trans. Amer. Math. Soc., 355(12):5054–5089, 2003.
- [14] K. JUNG, *Strongly 1-bounded von neumann algebras*, Geom. Func. Anal., 17 (4):1180–1200, 2007.

- [15] K. JUNG, *A hyperfinite inequality for free entropy dimension*, Proc. Amer. Math. Soc., 134(7):2099–2108 (electronic), 2006.
- [16] C. PEARCY, *W^* -algebras with a single generator*, Proc. Amer. Math. Soc., 13:831–832, 1962.
- [17] F. RĂDULESCU, *The fundamental group of the von Neumann algebra of a free group with infinitely many generators is \mathbb{R}_+* , J. Amer. Math. Soc., 5(3):517–532, 1992.
- [18] F. RĂDULESCU, *Random matrices, amalgamated free products and subfactors of the von Neumann algebra of a free group, of noninteger index*, Invent. Math., 115(2):347–389, 1994.
- [19] J. SHEN, *Singly generated II_1 factors*, J. Operator Theory, to appear. arXiv:math.OA/0511327, 2005.
- [20] A. M. SINCLAIR AND R. R. SMITH, *Finite von Neumann algebras and masas*, Volume 351 of London Mathematical Society Lecture Notes Series, Cambridge University Press, Cambridge, 2008.
- [21] S. J. SZAREK, *Nets of Grassmann manifold and orthogonal group*, In Proceedings of research workshop on Banach space theory (Iowa City, Iowa, 1981), pages 169–185, Iowa City, IA, 1982. Univ. Iowa.
- [22] D. M. TOPPING, *Lectures on von Neumann algebras*, Number 36 in Van Nostrand Reinhold Mathematical Studies. Van Nostrand, London, 1971.
- [23] D. VOICULESCU, *The analogues of entropy and of Fisher’s information measure in free probability theory, II*, Invent. Math., 118(3):411–440, 1994.
- [24] D. VOICULESCU, *The analogues of entropy and of Fisher’s information measure in free probability theory, III, The absence of Cartan subalgebras*, Geom. Funct. Anal., 6(1):172–199, 1996.
- [25] D. VOICULESCU, K. J. DYKEMA, AND A. NICA, *Free Random Variables*, volume 1 of CRM Monograph, American Mathematical Society, Providence, 1992.
- [26] W. WOGEN, *On generators for von Neumann algebras*, Bull. Amer. Math. Soc., 75:95–99, 1969.