ON A REDUCTION PROCEDURE FOR HORN
INEQUALITIES IN FINITE VON NEUMANN ALGEBRAS

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Abstract. We consider the analogues of the Horn inequalities in finite von Neumann algebras, which concern the possible spectral distributions of sums $a + b$ of self–adjoint elements a and b in a finite von Neumann algebra. It is an open question whether all of these Horn inequalities must hold in all finite von Neumann algebras, and this is related to Connes' embedding problem. For each choice of integers $1 \leqslant r \leqslant n$, there is a set T_r^n of Horn triples (I, J, K) of r–tuples of integers, and the Horn inequalities are in one–to–one correspondence with $\cup_{1 \leqslant r \leqslant n} T_r^n$.

We consider a property P_n, analogous to one introduced by Therianos and Thompson in the case of matrices, amounting to the existence of projections having certain properties relative to arbitrary flags, which guarantees that a given Horn inequality holds in all finite von Neumann algebras. It is an open question whether all Horn triples in T_r^n have property P_n. Certain triples in T_r^n can be reduced to triples in T_r^{n-1} by an operation we call *TT–reduction*. We show that property P_n holds for the original triple if property P_{n-1} holds for the reduced one.

A major part of this paper is devoted to showing that this operation of reduction preserves the value of the corresponding Littlewood–Richardson coefficients. We then characterize the TT–irreducible Horn triples in T_3^n, for arbitrary n, and for those LR–minimal ones (namely, those having Littlewood–Richardson coefficient equal to 1), we perform a construction of projections with respect to flags in arbitrary von Neumann algebras in order to prove property P_n for them. This shows that all LR–minimal triples in $\cup_{n \geqslant 3} T_3^n$ have property P_n, and so that the corresponding Horn inequalities hold in all finite von Neumann algebras.

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