Operators
and
Matrices
Volume 3, Number 2 (2009), 241–260

# TRACE INEQUALITIES AND SPECTRAL SHIFT

ANNA SKRIPKA

*Abstract.* We derive monotonicity and convexity inequalities for traces of operator functions defined on self-adjoint elements of a semi-finite von Neumann algebra. Among tools involved in the proofs are a generalized Birman-Solomyak spectral averaging formula (obtained in the paper), a generalized Birman-Schwinger principle, and Koplienko's spectral shift function, a new, more straightforward, approach to which is developed in the paper.

*Mathematics subject classification* (2000): Primary 26A48, 26A51, 47A56; Secondary 47A55, 47C15.
*Keywords and phrases*: monotonicity, convexity, von Neumann algebra, spectral shift function, operator function.

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Paper OaM-03-15

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