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TRACE INEQUALITIES AND SPECTRAL SHIFT

ANNA SKRIPKA

Abstract. We derive monotonicity and convexity inequalities for traces of operator functions defined on self-adjoint elements of a semi-finite von Neumann algebra. Among tools involved in the proofs are a generalized Birman-Solomyak spectral averaging formula (obtained in the paper), a generalized Birman-Schwinger principle, and Koplienko's spectral shift function, a new, more straightforward, approach to which is developed in the paper.

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REFERENCES

[1] N. A. AZAMOV, A. L. CAREY, P. G. DODDS, F. A. SUKOCHEV, *Operator integrals, spectral shift, and spectral flow,* Canad. J. Math., to appear.
[2] N. A. AZAMOV, P. G. DODDS, F. A. SUKOCHEV, *The Krein spectral shift function in semifinite von Neumann algebras*, Integr. Equ. Oper. Theory, **55** (2006), 347–362.
[3] M. SH. BIRMAN, M. Z. SOLOMYAK, *Remarks on the spectral shift function,* Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **27** (1972), 3–26; English transl. in J. Soviet Math. **3**, 4 (1975), 408–419.
[4] K. N. BOYADZHIEV, *Mean value theorems for traces,* Math. Japon. **38** (1993), 217–224.
[5] L. G. BROWN, H. KOSAKI, *Jensen's inequality in semi-finite von Neumann algebras,* J. Operator Theory, **23** (1990), 3–29.
[6] R. W. CAREY, J. D. PINCUS, *Mosaics, principal functions, and mean motion in von Neumann algebras*, Acta Math., **138** (1977), 153–218.
[7] J. Dixmier, *Von Neumann Algebras*, North-Holland, Amsterdam, 1981.
[8] T. FACK, *Sur la notion de valeur caractéristique,* J. Operator Theory, **7** (1982), 307–333.
[9] T. FACK, H. KOSAKI, *Generalized s-numbers of τ-measurable operators,* Pacific J. Math., **123**, 2 (1986), 269–300.
[10] F. GESZTESY, K. A. MAKAROV, A. K. MOTOVILOV, *Monotonicity and concavity properties of the spectral shift function.* Stochastic processes, physics and geometry: new interplays, II (Leipzig, 1999), 207–222, CMS Conf. Proc., **29**, Amer. Math. Soc., Providence, RI, 2000.
[11] F. GESZTESY, K. A. MAKAROV, S. N. NABOKO, *The spectral shift operator,* in J. Dittrich, P. Exner, and M. Tater (eds.) "Mathematical Results in Quantum Mechanics", Operator Theory: Advances and Applications, Vol. 108, Birkhäuser, Basel, 1999, 59–90.
[12] F. GESZTESY, A. PUSHNITSKI, B. SIMON, *On the Koplienko Spectral shift function, I. Basics,* Zh. Mat. Fiz. Anal. Geom., **4**, 1 (2008), 63–107.
[13] F. HANSEN, G. K. PEDERSEN, *Jensen's operator inequality,* Bull. London Math. Soc., **35** (2003), 553–564.
[14] B. FUGLEDE AND R. V. KADISON, *Determinant theory in finite factors*, Ann. Math., **55** (1952), 520–530.
[15] R. V. KADISON AND J. R. RINGROSE, *Fundamentals of the Theory of Operator Algebras Vol. II,* Academic Press, Orlando, FL, 1986.
[16] L. S. KOPLIENKO, *Trace formula for nontrace-class perturbations,* Sibirsk. Mat. Zh., **25**, 5 (1984), 6–21 (Russian). English translation: Siberian Math. J., **25**, 5 (1984), 735–743.

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[17] V. KOSTRYKIN, *Concavity of eigenvalue sums and the spectral shift function,* J. Funct. Anal., **176**, 1 (2000), 100–114.

[18] V. KOSTRYKIN, K. A. MAKAROV, A. SKRIPKA, *The Birman-Schwinger principle in von Neumann algebras of finite type,* J. Funct. Anal., **247** (2007), 492–508.

[19] M. G. KREIN, *On a trace formula in perturbation theory*, Matem. Sbornik, **33** (1953), 597–626 (Russian).

[20] E. H. LIEB, H. SIEDENTOP, *Convexity and concavity of eigenvalue sums,* J. Statist. Phys., **63** (1991), 811–816.

[21] I. M. LIFSHITS, *On a problem of the theory of perturbations connected with quantum statistics*, Uspehi Matem. Nauk, **7** (1952), 171–180 (Russian).

[22] K. A. MAKAROV, A. SKRIPKA, *Some applications of the perturbation determinant in finite von Neumann algebras,* Canad. J. Math., to appear.

[23] H. NEIDHARDT, *Spectral shift function and Hilbert-Schmidt perturbation: extensions of some work of L.S. Koplienko,* Math. Nachr., **138** (1988), 7–25.

[24] V.V. PELLER, *Hankel operators in the perturbation theory of unbounded self-adjoint operators.* Analysis and partial differential equations, Lecture Notes in Pure and Applied Mathematics, 122, Dekker, New York, 1990, 529–544.

[25] V. V. PELLER, *An extension of the Koplienko-Neidhardt trace formulae,* J. Funct. Anal., **221** (2005), 456–481.

[26] V. V. PELLER, *Multiple operator integrals and higher operator derivatives,* J. Funct. Anal., **233** (2006), 515–544.

[27] D. PETZ, *Spectral scale of self-adjoint operators and trace inequalities,* J. Math. Anal. Appl., **109** (1985), 74–82.

[28] M. REED, B. SIMON, *Methods of Modern Mathematical Physics I, Functional Analysis,* 2nd. ed., Academic Press, New York, 1980.

[29] D. RUELLE, *Statistical Mechanics. Rigorous Results,* Bejamin, New York, 1969.

[30] S. SAKAI, *C^*-algebras and W^*-algebras,* Ergebn. Math. und ihrer Grenzgeb., 60, Springer-Verlag, New York-Heilderberg, 1971.

[31] M. A. SHUBIN, *Discrete magnetic Laplacian,* Comm. Math. Phys. **164**, 2 (1994), 259–275.

[32] B. SIMON, *Spectral averaging and the Krein spectral shift,* Proc. Amer. Math. Soc., **126**, 5 (1998), 1409–1413.

[33] A. SKRIPKA, *On properties of the ξ-function in semi-finite von Neumann algebras,* Integr. Equ. Oper. Theory, **62**, 2 (2008), 247–267.

[34] M. TAKESAKI, *Theory of operator algebras. I,* Springer-Verlag, New York-Heidelberg, 1979.

[35] D. VOICULESCU, *On a trace formula of M. G. Krein,* Operator Theory: Adv. Appl., **24** (1987), 329–332.