

MATRICES WITH NORMAL DEFECT ONE

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Abstract. A $n \times n$ matrix A has normal defect one if it is not normal, however can be embedded as a north-western block into a normal matrix of size $(n+1) \times (n+1)$. The latter is called a minimal normal completion of A . A construction of all matrices with normal defect one is given. Also, a simple procedure is presented which allows one to check whether a given matrix has normal defect one, and if this is the case — to construct all its minimal normal completions. A characterization of the generic case for each n under the assumption $\text{rank}(A^*A - AA^*) = 2$ (which is necessary for A to have normal defect one) is obtained. Both the complex and the real cases are considered. It is pointed out how these results can be used to solve the minimal commuting completion problem in the classes of pairs of $n \times n$ Hermitian (resp., symmetric, or symmetric/antisymmetric) matrices when the completed matrices are sought of size $(n+1) \times (n+1)$. An application to the $2 \times n$ separability problem in quantum computing is described.

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