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NORMALS, SUBNORMALS AND AN OPEN QUESTION

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Abstract. An acute look at basic facts concerning unbounded subnormal operators is taken here. These operators have the richest structure and are the most exciting among the whole family of beneficiaries of the normal ones. Therefore, the latter must necessarily be taken into account as the reference point for any exposition of subnormality. So as to make the presentation more appealing a kind of comparative survey of the bounded and unbounded case has been set forth.

This piece of writing serves rather as a practical guide to this largely impenetrable territory than an exhausting report.

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