Operators and Matrices
Volume 5, Number 1 (2011), 41–52

MULTIPLICITIES, BOUNDARY POINTS, AND JOINT NUMERICAL RANGES

WAI-SHUN CHEUNG, XUHUA LIU AND TIN-YAU TAM

Abstract. The multiplicity of a point in the joint numerical range $W(A_1,A_2,A_3) \subseteq \mathbb{R}^3$ is studied for $n \times n$ Hermitian matrices A_1,A_2,A_3. The relative interior points of $W(A_1,A_2,A_3)$ have multiplicity greater than or equal to $n-2$. The lower bound $n-2$ is best possible. Extreme points and sharp points are studied. Similar study is given to the convex set $V(A) := \{x^T Ax : x \in \mathbb{R}^n, x^T x = 1\} \subseteq \mathbb{C}$, where $A \in \mathbb{C}_{n\times n}$ is symmetric. Examples are given.

Mathematics subject classification (2010): Primary: 15A60.
Keywords and phrases: Joint numerical range, multiplicity, extreme point, sharp point, boundary point.

REFERENCES

[1] Y. H. AU-YEUNG AND Y. T. POON, *A remark on the convexity and positive definiteness concerning Hermitian matrices*, Southeast Asian Bull. Math., **3** (1979), 85–92.
[2] Y. H. AU-YEUNG AND N. K. TSING, *An extension of the Hausdorff-Toeplitz theorem*, Proc. Amer. Math. Soc., **89** (1983), 215–218.
[3] Y. H. AU-YEUNG AND N. K. TSING, *Some theorems on the numerical range*, Linear and Multilinear Algebra, **15** (1984), 215–218.
[4] P. L. BINDING AND C. K. LI, *Joint ranges of Hermitian matrices and simultaneous diagonalization*, Linear Algebra Appl., **151** (1991), 157–168.
[5] L. BRICKMAN, *On the field of values of a matrix*, Proc. Amer. Math. Soc., **12** (1961), 61–66.
[6] C. DAVIS, *The Toeplitz-Hausdorff theorem explained*, Canad. Math. Bull., **14** (1971), 245–246.
[7] W. F. DONOGHUE, JR., *On the numerical range of a bounded operator*, Michigan Math. J., **4** (1957), 261–263.
[8] M. R. EMBRY, *The numerical range of an operator*, Pacific J. Math., **32** (1970), 647–650.
[9] R. A. HORN AND C. R. JOHNSON, *Topics in Matrix Analysis*, Cambridge University Press, 1991.
[10] C. K. LI AND Y. T. POON, *Convexity of the joint numerical range*, SIAM J. Matrix Anal. Appl., **21** (1999), 668–678.
[11] C. K. LI AND T. Y. TAM, *Numerical ranges arising from simple Lie algebras*, J. Canad. Math. Soc., **52** (2000), 141–171.
[12] A. MCINTOSH, *The Toeplitz-Hausdorff theorem and ellipticity conditions*, Amer. Math. Monthly, **85** (1978), 475–477.
[13] Y. T. POON, *Generalized numerical ranges, joint positive definiteness and multiple eigenvalues*, Proc. Amer. Math. Soc., **125** (1997), 1625–1634.
[14] J. G. STAMPFLI, *Extreme points of the numerical range of hyponormal operators*, Michigan Math. J., **13** (1966), 87–89.
[15] F. UHLIG, *An inverse field of values problem*, Inverse Problems, **24** (2008), 055019, 19 pages.