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POSITIVE LINEAR FUNCTIONALS WITHOUT REPRESENTING MEASURES

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Abstract. For k even, let $\mathscr{P}_k$ denote the vector space of polynomials in 2 real variables of degree at most k. A linear functional $L : \mathscr{P}_k \longrightarrow \mathbb{R}$ is *positive* if $p \in \mathscr{P}_k,\ p|\mathbb{R}^2 \geqslant 0 \Longrightarrow L(p) \geqslant 0$. Hilbert's theorem on sums of squares (cf. [15]) implies that $L : \mathscr{P}_4 \longrightarrow \mathbb{R}$ is positive if and only if the moment matrix associated to L is positive semidefinite. In this note, using $k = 6$, we exhibit the first family of positive linear functionals $L : \mathscr{P}_k \to \mathbb{R}$ whose positivity cannot be derived from the positive semidefiniteness of the associated moment matrices, and which do not correspond to integration with respect to positive measures.

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REFERENCES

[1] C. BAYER AND J. TEICHMANN, *The proof of Tchakaloff's Theorem*, Proc. Amer. Math. Soc., **134** (2006), 3035–3040.
[2] S. BERBERIAN, *Lectures in Functional Analysis and Operator Theory*, Springer-Verlag, 1973.
[3] R. CURTO AND L. FIALKOW, *Recursiveness, positivity, and truncated moment problems*, Houston J. Math. **17** (1991) 603–635.
[4] R. CURTO AND L. FIALKOW, *Solution of the truncated complex moment problem for flat data*, Memoirs of the American Mathematical Society, **119** (1996), No. 568, Amer. Math. Soc. Providence, RI, 1996.
[5] R. CURTO AND L. FIALKOW, *Flat extensions of positive moment matrices: Relations in analytic or conjugate terms*, Operator Th.: Adv. Appl. **104** (1998), 59–82.
[6] R. CURTO AND L. FIALKOW, *Solution of the singular quartic moment problem*, Journal of Operator Theory, **48** (2002), 315–354.
[7] R. CURTO AND L. FIALKOW, *Truncated K-moment problems in several variables*, Journal of Operator Theory, **54** (2005), 189–226.
[8] R. CURTO AND L. FIALKOW, *Solution of the truncated hyperbolic moment problem*, Integral Equations and Operator Theory, **52** (2005), 181–219.
[9] R. CURTO AND L. FIALKOW, *An analogue of the Riesz-Haviland Theorem for the truncated moment problem*, J. Functional Analysis, **225** (2008), 2709–2731.
[10] L. FIALKOW, *Truncated multivariable moment problems with finite variety*, J. Operator Theory, **60** (2008), 343–377.
[11] L. FIALKOW, *Solution of the truncated moment problem with variety* $y = x^3$, Trans. Amer. Math. Soc., **363** (2011), 3133–3165.
[12] L. FIALKOW AND J. NIE, *Positivity of Riesz functionals and solutions of quadratic and quartic moment problems*, J. Functional Analysis, **258**, 1 (2010), 328–356.
[13] E. K. HAVILAND, *On the momentum problem for distributions in more than one dimension II*, Amer. J. Math., **58** (1936) 164–168.
[14] M. LAURENT, *Sums of squares, moment matrices and optimization over polynomials, Emerging Applications of Algebraic Geometry*, Vol. 149 of IMA Volumes in Mathematics and its Applications, M. Putinar and S. Sullivant (eds), Springer, pages 157–270, 2009.

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[15] B. REZNICK, *Some concrete aspects of Hilbert's* 17^{th} *problem*, in Contemp. Math., volume 253, pages 251–272, American Mathematical Society, 2000.
[16] B. REZNICK, *On Hilbert's construction of positive polynomials*, preprint, 2007.
[17] M. RIESZ, *Sur le probleme des moments, troisieme note*, Ark. Mat., **17** (1923), 1–52.
[18] K. SCHMÜDGEN, *An example of a positive polynomial which is not a sum of squares of polynomials. A positive, but not strongly positive functional*, Math. Nachr., **88** (1979), 385–390.
[19] K. SCHMÜDGEN, *The K-moment problem for compact semi-algebraic sets*, Math. Ann. **289** (1991), 203–206.
[20] V. TCHAKALOFF, *Formules de cubatures mécanique à coefficients non négatifs*, Bull. Sci. Math., **(2)82** (1957), 123–134.