

THE SIMILARITY PROBLEM FOR INDEFINITE STURM—LIOUVILLE OPERATORS WITH PERIODIC COEFFICIENTS

ALEKSEY KOSTENKO

Abstract. We investigate the problem of similarity to a self-adjoint operator for J -positive Sturm–Liouville operators $L = \frac{1}{\omega} \left(-\frac{d^2}{dx^2} + q \right)$ with 2π -periodic coefficients q and ω . It is shown that if 0 is a critical point of the operator L , then it is a singular critical point. This gives us a new class of J -positive differential operators with the singular critical point 0. Also, we extend the Beals and Parfenov regularity conditions for the critical point ∞ to the case of operators with periodic coefficients.

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