

## PARA-ORTHOGONAL RATIONAL MATRIX-VALUED FUNCTIONS ON THE UNIT CIRCLE

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*Abstract.* In this paper, we continue previous investigations with the ultimate goal being a Szegő theory for orthogonal rational matrix functions. We implement here the concept of para-orthogonal functions on the unit circle in the context of rational matrix functions and present some fundamental properties of the para-orthogonal functions in question. We discuss, among other things, the relationship between these functions and orthogonal rational matrix functions as well as existence criteria and some para-orthogonal functions of particular interest.

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