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# UNIFORMLY $\gamma$—RADONIFYING FAMILIES OF OPERATORS AND THE STOCHASTIC WEISS CONJECTURE

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*Abstract.* We introduce the notion of uniform $\gamma$–radonification of a family of operators, which unifies the notions of $R$–boundedness of a family of operators and $\gamma$–radonification of an individual operator. We study the properties of uniformly $\gamma$–radonifying families of operators in detail and apply our results to the stochastic abstract Cauchy problem

$$dU(t) = AU(t)\,dt + B\,dW(t), \quad U(0) = 0.$$

Here, $A$ is the generator of a strongly continuous semigroup of operators on a Banach space $E$, $B$ is a bounded linear operator from a separable Hilbert space $H$ into $E$, and $W_H$ is an $H$–cylindrical Brownian motion. When $A$ and $B$ are simultaneously diagonalisable, we prove that an invariant measure exists if and only if the family

$$\{\sqrt{\lambda}R(\lambda,A)B : \lambda \in S_\vartheta\}$$

is uniformly $\gamma$–radonifying for some/all $0 < \vartheta < \frac{\pi}{2}$, where $S_\vartheta$ is the open sector of angle $\vartheta$ in the complex plane. This result can be viewed as a partial solution of a stochastic version of the Weiss conjecture in linear systems theory.

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