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ERROR REPRESENTATION FORMULA FOR EIGENVALUE APPROXIMATIONS FOR POSITIVE DEFINITE OPERATORS

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Abstract. The main contribution of this paper is an error representation formula for eigenvalue approximations for positive definite operators defined as quadratic forms. The formula gives an operator theoretic framework for treating discrete eigenvalue approximation/estimation problems for unbounded positive definite operators independent of the multiplicity. Furthermore, by the use of the error representation formula, we give computable lower and upper estimates for discrete eigenvalues of such operators. The estimates could be seen as being of the Kato–Temple type. Our estimates can be applied to the Rayleigh–Ritz approximation on the test subspace which is a subset of the corresponding form domain of the operator. We present several completely soluble prototype examples for an application of the presented theory and argue the optimality of our approach in this context.

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