

## WEYL THEORY AND EXPLICIT SOLUTIONS OF DIRECT AND INVERSE PROBLEMS FOR DIRAC SYSTEM WITH A RECTANGULAR MATRIX POTENTIAL

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*Abstract.* A non-classical Weyl theory is developed for Dirac systems with rectangular matrix potentials. The notion of the Weyl function is introduced and the corresponding direct problem is solved. Furthermore, explicit solutions of the direct and inverse problems are obtained for the case of rational Weyl matrix functions.

*Mathematics subject classification (2010):* 34B20, 34L40, 15A15, 93B15.

*Keywords and phrases:* Weyl function, Weyl theory, Dirac system, rectangular matrix potential, direct problem, inverse problem, pseudo-exponential potential, explicit solution, rational matrix function, non-expansive matrix function, realization.

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