Operators
and
Matrices
Volume 7, Number 2 (2013), 241–283
doi:10.7153/oam-07-15

INITIAL VALUE PROBLEMS AND WEYL–TITCHMARSH THEORY FOR SCHRÖDINGER OPERATORS WITH OPERATOR–VALUED POTENTIALS

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Abstract. We develop Weyl–Titchmarsh theory for self-adjoint Schrödinger operators H_α in $L^2((a,b);dx;\mathscr{H})$ associated with the operator-valued differential expression $\tau = -(d^2/dx^2) + V(\cdot)$, with $V:(a,b)\to\mathscr{B}(\mathscr{H})$, and $\mathscr{H}$ a complex, separable Hilbert space. We assume regularity of the left endpoint a and the limit point case at the right endpoint b. In addition, the bounded self-adjoint operator $\alpha=\alpha^*\in\mathscr{B}(\mathscr{H})$ is used to parametrize the self-adjoint boundary condition at the left endpoint a of the type

$$\sin(\alpha)u'(a)+\cos(\alpha)u(a)=0,$$

with u lying in the domain of the underlying maximal operator $H_{\max}$ in $L^2((a,b);dx;\mathscr{H})$ associated with τ. More precisely, we establish the existence of the Weyl–Titchmarsh solution of H_α, the corresponding Weyl–Titchmarsh m-function m_α and its Herglotz property, and determine the structure of the Green's function of H_α.

Developing Weyl–Titchmarsh theory requires control over certain (operator-valued) solutions of appropriate initial value problems. Thus, we consider existence and uniqueness of solutions of 2nd-order differential equations with the operator coefficient V,

$$\begin{cases} -y''+(V-z)y=f \text{ on } (a,b), \\ y(x_0)=h_0,\ y'(x_0)=h_1, \end{cases}$$

under the following general assumptions: $(a,b)\subseteq\mathbb{R}$ is a finite or infinite interval, $x_0\in(a,b)$, $z\in\mathbb{C}$, $V:(a,b)\to\mathscr{B}(\mathscr{H})$ is a weakly measurable operator-valued function with $\|V(\cdot)\|_{\mathscr{B}(\mathscr{H})}\in L^1_{\rm loc}((a,b);dx)$, and $f\in L^1_{\rm loc}((a,b);dx;\mathscr{H})$. We also study the analog of this initial value problem with y and f replaced by operator-valued functions $Y,F\in\mathscr{B}(\mathscr{H})$.

Our hypotheses on the local behavior of V appear to be the most general ones to date.

Mathematics subject classification (2010): Primary: 34A12, 34B20, 34B24; Secondary: 47E05.
Keywords and phrases: Weyl–Titchmarsh theory, ODEs with operator coefficients, Schrödinger operators.

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