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ON THE SPECTRA AND PSEUDOSPECTRA OF A CLASS OF NON–SELF–ADJOINT RANDOM MATRICES AND OPERATORS

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Abstract. In this paper we develop and apply methods for the spectral analysis of non-self-adjoint tridiagonal infinite and finite random matrices, and for the spectral analysis of analogous deterministic matrices which are pseudo-ergodic in the sense of E. B. Davies (Commun. Math. Phys. 216 (2001), 687–704). As a major application to illustrate our methods we focus on the "hopping sign model" introduced by J. Feinberg and A. Zee (Phys. Rev. E 59 (1999), 6433–6443), in which the main objects of study are random tridiagonal matrices which have zeros on the main diagonal and random ± 1's as the other entries. We explore the relationship between spectral sets in the finite and infinite matrix cases, and between the semi-infinite and bi-infinite matrix cases, for example showing that the numerical range and p-norm ε-pseudospectra ($\varepsilon > 0$, $p \in [1,\infty]$) of the random finite matrices converge almost surely to their infinite matrix counterparts, and that the finite matrix spectra are contained in the infinite matrix spectrum Σ. We also propose a sequence of inclusion sets for Σ which we show is convergent to Σ, with the nth element of the sequence computable by calculating smallest singular values of (large numbers of) $n \times n$ matrices. We propose similar convergent approximations for the 2-norm ε-pseudospectra of the infinite random matrices, these approximations sandwiching the infinite matrix pseudospectra from above and below.

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