Operators and Matrices
Volume 7, Number 4 (2013), 777–811
doi:10.7153/oam-07-44

INDEFINITE STURM–LIOUVILLE OPERATORS WITH PERIODIC COEFFICIENTS

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Abstract. We investigate the spectral properties of the maximal operator A associated with a differential expression $\frac{1}{w}\left(-\frac{d}{dx}\left(p\frac{d}{dx}\right)+q\right)$, where the coefficients w, p and q are real-valued and w changes sign. It turns out that the non-real spectrum of A is bounded, symmetric with respect to the real axis and consists of a finite number of analytic curves. The real spectrum is band-shaped and neither bounded from above nor from below. We characterize the finite spectral singularities of A and prove that there is only a finite number of them. Finally, we provide a condition on the coefficients w and p which ensures that ∞ is not a spectral singularity of A.

Mathematics subject classification (2010): 34B09, 34B24, 34L10, 47B50, 47E05.
Keywords and phrases: Sturm-Liouville operator, periodic coefficients, indefinite weight.

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